



Ruler by Steps (RbySteps): Exploring Length Measurement with Young Children Through a Sequential Ruler Construction

Ariadna Gómezescobar¹

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Abstract

Measuring lengths with a ruler poses a challenging task for young children, who often struggle with issues related to identifying the unit on the measuring instrument and interpreting the starting point of the measurement. This study introduces RbySteps, a didactic sequencing proposal designed to address these difficulties by simulating the step-by-step construction of a ruler, significantly including the elements of the ruler: numbers, marks and units. A total of 36 children enrolled in the final year of early childhood education (5–6 years old) participated in the study, of whom 19 were assigned to an experimental group and utilized RbySteps, while the other 17, constituting the control group, followed a conventional length teaching format. Following the intervention, the experimental group demonstrated enhanced proficiency in measuring lengths with a ruler, whereas the control group showed no similar improvement. This leads to the proposal of a Hypothetical Learning Trajectory (HLT) for length measurement with a ruler based on RbySteps.

Keywords Measurement · Length · Ruler · Didactic proposal · Learning trajectory · Childhood education

The tangible realm of physical objects and the conceptual dimension of numbers share a connection, primarily through magnitudes and measurement, among other aspects. Understanding these concepts is significant not only in fostering the advancement of mathematics and experimental sciences but also in the context of daily life. Specifically, the concept of length serves as the foundation for other magnitudes, such as area or volume, underscoring its pivotal role in the learning process. However, during early education, the emphasis appears to lean towards shapes and numerical counting, sometimes at the expense of the understanding of measurements (Hindman, 2013; Sarama & DiBiase, 2004).

In addition, notwithstanding the apparent familiarity with measurement, numerous studies highlight the challenges that young children encounter when using rulers (Bragg & Outhred, 2004; Congdon et al., 2018; Cullen & Barrett, 2010; Gómezescobar et al., 2018, 2023; Irwin et al., 2004; Levine et al., 2009; Sisman & Aksu, 2016; Solomon

et al., 2015). To address these challenges, particularly in terms of identifying the unit in the measuring instrument and interpreting the starting point of measurement, this study introduces RbySteps. This article aims to elucidate the significance of measurement units on the ruler, facilitating the transition from discrete units to the accurate use of the ruler—a tool in which length is depicted as a continuous magnitude.

The current study thus has two aims. The first is to evaluate through an exploratory quasi-experimental study, the effectiveness of an instructional proposal for teaching measurement with a ruler, which we call RbySteps. The second aim, derived from the first, is to propose a Hypothetical Learning Trajectory (HLT) to guide the teaching-learning process of ruler-based measurement.

Challenges in Measuring with Rulers

The ruler is a tool that portrays discrete units separated by marks, with numbers underneath indicating the “distance accumulation” up to a given point, a concept elucidated by Stephan and Clements (2003). While the interpretation of these elements (units, marks, and numbers) may initially appear straightforward, children often encounter specific

✉ Ariadna Gómezescobar
ariadna.gomezescobar@uclm.es

¹ Departamento de Matemáticas (Área de Didáctica), Facultad de Educación de Toledo, Universidad de Castilla-La Mancha, Av. de Carlos III, Toledo 45004, Spain

challenges, which are particularly evident in tasks involving non-simple measurement situations.

Non-simple measuring situations refer to instances where one end of the object being measured does not align with the zero mark on the ruler. This category of task requires comprehension of breaking down a continuous quantity into multiple equal discrete quantities (Solomon et al., 2015) and serves as a means to identify misconceptions in the conceptualization of measurements (Congdon et al., 2018).

The aforementioned studies and the thesis conducted by Gómezescobar (2020) reveal that the main difficulties children have in using rulers lie in the interpretation of two aspects: the units on the ruler and the starting point for measuring. These challenges are evident through the strategies employed by pupils, which can be broadly categorized into three main types: reading the number coinciding with the end of the object or reading the number at the endpoint (EP), counting marks (CM), and engaging with the interval (INT).

The effectiveness of the first strategy, reading the number at the endpoint (EP), is contingent upon placing the object at the zero mark beforehand or accurately interpreting the starting point and the endpoint. Several studies (Bragg & Outhred, 2004; Cullen & Barrett, 2010; Gómezescobar et al., 2018, 2023; Levine et al., 2009; Nunes, Ligth, & Mason, 1993; Sisman & Aksu, 2016; Solomon et al., 2015) report that when children encounter tasks involving shifted objects or broken rulers (non-simple measurement situations), those relying on the EP strategy often provide incorrect answers. It appears that these pupils focus on the numerical value without attributing any meaningful significance to it and fail to recognize the importance of the starting point in their measurements.

Meanwhile, some pupils believe that measuring simply involves counting, leading them to count the marks (CM) on the ruler, to which they assign the status of units (Gómezescobar, 2020). Consequently, these children struggle to properly interpret the units indicated on the ruler.

Lastly, the most conceptually sophisticated strategy in terms of measurement involves acting on the spatial interval or unit, along with its subsequent counting (INT). This approach demonstrates a keen understanding of the unit within the measuring instrument.

Teaching Ruler Measurement

The traditional approach to teaching measurement in both curriculum and instructional practices has historically emphasized the arithmetic aspects, such as base 10 conversion (Lee & Smith, 2011; Smith & Barrett, 2017), rather than placing a strong emphasis on comprehending the conceptual framework and procedural aspects of measurement.

In the context of length, understanding units entails recognizing them within the continuous attribute of an object.

Practical instruction for ruler measurements often focuses on the procedural steps, instructing pupils to place one edge of the object on the zero mark and read the number aligned with the right end of the ruler. This purely mechanical approach leads to some children completing primary school without acquiring the ability to identify the unit of measure for length on a ruler (Bragg & Outhred, 2004).

In many cases, the teaching of length measurement abruptly transitions from using discrete units to employing a ruler, with insufficient connection between these tasks (Levine et al., 2009). Consequently, students perceive these activities as entirely independent. To address this issue, studies recommend facilitating a smooth transition from measurement with discrete units to measurement with an instrument that presents length continuously, such as the ruler (Irwin et al., 2004; Levine et al., 2009; Gómezescobar, 2020). However, it appears that both curriculum design and sequencing proposals in the teaching of measuring magnitude, potentially due to their broad nature, do not adequately emphasize this essential transition (Alsina, 2006; Belmonte, 2005; Barrett et al., 2017; Clarke et al., 2003; Van den Heuvel-Panhuizen & Buys, 2008).

To facilitate this transition, some studies suggest introducing the gradual construction of the length measurement instrument, the ruler, in teaching (Bragg & Outhred, 2004; Clements & Stephan, 2004). This recommendation is further elaborated in the following section through the use of RbySteps.

RbySteps: A Didactic Sequence Proposal for Teaching Measurement with a Ruler

Recognizing the critical importance of accurately interpreting the components of a ruler (numbers, marks, and intervals), of understanding the starting point of measurements, and of facilitating the transition from discrete units to measurement with a ruler, Gómezescobar et al. (2019) propose a sequenced approach consisting of four ad-hoc rulers for teaching ruler measurement. In the limitations and improvements section of her thesis's, Gómezescobar (2020) suggests incorporating two additional rulers to complete the sequence. This article details the entire written sequence: RbySteps, a didactic proposal registered at the United States Copyright Office with registration number "TXu 2-427-014".

RbySteps may be included in the framework of Hypothetical Learning Trajectories (HLT), which have become increasingly prevalent in the teaching and learning of mathematical concepts. These trajectories establish a developmental progression for pupils and instructional tasks

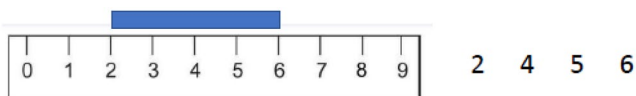


Fig. 1 Task M1, displaced object with response option

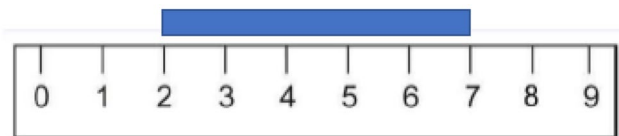


Fig. 2 Task M5, displaced object with free response

for teachers, guiding the path to achieve specific learning objectives. In the context of how children learn to measure length with a ruler, previous research (Gómezescobar, 2018, 2020; Levine et al., 2009; Solomon et al., 2015) underscores the challenge children face in interpreting the starting point of measurements and understanding the unit on the measuring instrument. This emphasizes the need for a specific transition between measurement with discrete objects and measurement with a ruler. This transition is addressed in RbySteps, the didactic proposal tested in this study, which gradually introduces the elements of the ruler in a simulated construction.

RbySteps sets out a step-by-step construction of a ruler comprised of seven distinct stages, which, if successfully piloted, could form an HLT, the feasibility of which might subsequently be explored. The didactic sequence is positioned within a broader developmental progression for the learning trajectory of length measurement, as defined and revised by Clements et al. (2017). Specifically, RbySteps focuses exclusively on length measurement with a ruler, starting at the “End-to-End Length Measurer” level where children use rulers with minimal guidance and some success, then moving through the “Length Unit Relater and Repeater” level characterized by measuring a length correctly with a ruler aligned at the zero mark, and ending at the “Consistent Length Measurer” level, where children can resolve broken-ruler tasks effectively. This last task is conceptually similar to the one suggested throughout the instructional sequence: placing an object offset on the ruler. The final stage of RbySteps aligns with Belmonte’s (2005) proposed last phase, involving measurements with the decimal metric system. RbySteps offers instructional guidelines for teaching measurement with a ruler, which can be found in the Procedure section.

Materials and Methods

The study presented in this article is exploratory and quasi-experimental. The research method has received ethical approval from the ethics committee at the Universidad Autónoma de Madrid (CEI-121-2444).

Sample

The sample was comprised of 39 children aged 5 to 6 years in the final year of early childhood education. This sample size aligns with those of reference studies in the field of length measurement (Barrett et al., 2012, 2017; Levine et al., 2009; Nunes et al., 1993; Sarama et al., 2021; Solomon et al., 2015). The choice of the last year of early childhood education was deliberate, given that these pupils had not received prior instruction in using a ruler in school. Additionally, previous studies indicate that this age group is capable of performing specific measurement tasks (Boulton-Lewis, 1987; Clements, 1999; Hiebert, 1981; Gómezescobar et al., 2023).

The entire sample attended the same school, with the experimental group consisting of 19 children from one classroom and the control group comprising 17 from another. This design is quasi-experimental, as the children were not randomly assigned to the control and experimental groups. Three children were excluded from the study—one with autism spectrum disorder and two due to curricular gaps.

Evaluation Instrument: Test

A written worksheet with 8 ruler measurement tasks was used as the assessment instrument to evaluate pupils’ knowledge of measuring length with rulers. The tasks included in the test are similar to those proposed in previous landmark studies (Bragg & Outhred, 2004; Cogndon et al. 2018; Cullen & Barrett, 2010; Irwin et al., 2004; Kamii, 1995; Levine et al., 2009; Solomon et al., 2015). The first 4 measurement tasks (M1, M2, M3 and M4) have response options (Fig. 1, M1), while the last 4 (M5, M6, M7 and M8) do not, with the response being free (Fig. 2, M5). In the first 4 tasks the question is “How long is it? Circle the result”, and in the last 4 it is “How long is it? Write the result”.

In addition, in order to understand the measurement strategy used, we followed the approach proposed in the study by Levine et al. (2009). Specifically, in tasks M1, M2, M3 and M4 the numerical options provided are designed to reflect the correct answer or the different strategies: EP represents the strategy where the participant reads the number at either end of the object, and CM represents the strategy where the participant counts the marks on the ruler. For example, for

Fig. 1, “4” would be the correct answer, “2” and “6” would correspond to the EP strategy, and “5” to the CM strategy.

Notably, in measurement tasks M4 and M8, the object is aligned with the zero mark on the ruler (Fig. 3), while in the remaining six measurements (M1, M2, M3, M5, M6, and M7), the object is deliberately displaced.

The test underwent multiple iterative revisions by both the teachers and the researcher. Adjustments were made to the written test format (using capital letters for the written question “HOW LONG IS IT?”), including a separation between tasks, the size of drawings of the rulers, and the arrangement of answers. Regarding the latter, a decision was made to position the answer options to the right of the ruler rather than below it. This change ensured that the numerical options did not coincide with the numbers on the ruler. Additionally, teachers were trained to assist the children with reading, numeration, and other related tasks to ensure they understood the questions and could participate effectively.

As for the reliability of the instrument, Cronbach’s alpha was used to measure the internal consistency of the test, obtaining a value of 0.83.

Procedure

This section is structured as follows. It begins by commenting on the guidelines for the administration of the pre-test, and then specify issues common to the interventions carried out. Next, the section delves into the interventions in the experimental group with RbySteps and in the control group. Finally, issues related to the post-test are reported.

Pre-Test

As previously mentioned in the characterization of the sample, no participant had been instructed on measuring length with a ruler in a school context. However, the children were familiar with the instrument, as they had rulers in the classroom and used them to draw straight lines. For pupils to focus on length measurement with a ruler and to avoid obstacles related to numeration (counting, writing, etc.) in the test, the teacher assisted each child in the pre-test.

The teachers were trained to refrain from intervening in the correction of the answers; additionally for the presentation of the tasks, they were encouraged to use formulas such as “how long is this strip?”, “if we were to inform a friend

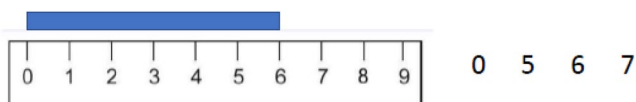


Fig. 3 Task M4, object aligned on zero mark with response option

of the length of this strip, what length should we tell them?” or “write down (or circle) the length here”.

In order to work on “sensory estimation”, the first phase for working with magnitudes as defined by Belmonte (2005), which focuses on the appreciation of the magnitude, the process started by asking the participant about their understanding of measurement and their previous experiences with it [“Can you tell me what you know about measuring things? Have you ever measured anything before?”]. The child was then instructed to run their fingers along a strip of cardboard to emphasize the concept of length [“Please run your fingers along this strip of cardboard. Can you feel how long it is?”]. Subsequently, the teacher administered the test outlined in the preceding section “Evaluation instrument: test”: pre-test. Importantly, the teachers refrained from indicating the correctness of measurements but provided assistance when pupils encountered challenges with numbering and/or counting. For example, if a child struggled with counting the marks on the ruler, the teacher might say, [“Let’s count the marks together. One, two, three...”] or if a child had difficulty writing the numbers, the teacher might assist by saying, [“Would you like me to help you write the number?”]

Intervention: General Guidelines

The pre-test was followed the next day by the intervention. The intervention in each group was implemented over two one-hour sessions on two consecutive days. In each session, the same intervention was repeated. To increase interaction during instruction, both in the experimental group and in the control group, four random subgroups were generated in each instruction session. Thus, the experimental group had three subgroups of 5 children and one subgroup of 4, while the control group was divided into three subgroups of 4 students and one subgroup of 5.

In all the subgroups, the intervention, conducted by one teacher, began by presenting a conventional ruler to the children. They were asked to express their thoughts on its purpose and characteristics. A consensus was reached that it was an instrument designed for measurement, primarily composed of numbers and marks.

Subsequently, the subgroups sat together and were given a few minutes to experiment with the material (rulers, units and cardboard strips for measuring). The intervention (experimental or control) guided by the teacher for approximately half an hour, was then conducted in each subgroup. During the intervention (experimental or control), the children measured several strips of cardboard aligned and displaced with respect to the zero mark on the ruler(s). In addition, at each measurement, the teacher systematically asked, “how much does it measure?” and “how did you know?”, so that the participants were aware of their measurement strategies.



Fig. 4 Measurement units

The last minutes of the intervention were used to measure strips of cardboard aligned and misaligned with the zero mark on the ruler(s).

Intervention in the Experimental Group

1. The teacher followed the seven-step RbySteps sequence, which simulates the sequential construction of a ruler, emphasizing the transition between discrete units and the ruler, a continuous measurement tool. The seven steps are as follows: Measurement of different objects using discrete units (Fig. 4), similar to those contained in the rulers of the set.
2. Under the pretext of minimizing the need for unit tessellation with each measurement, the proposal suggests presenting units pre-tiled on a support, resulting in the creation of the first ruler in the set (see Fig. 5). This ruler, along with the subsequent ones, underscores the significance of measuring objects aligned and displaced from the zero mark.



Fig. 5 Ruler with tiled units



Fig. 6 Ruler with tiled units delimited by marks

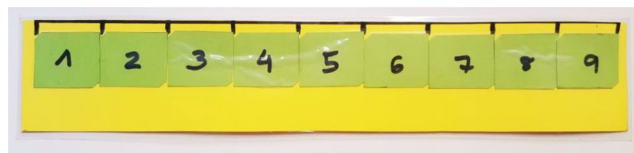


Fig. 7 Ruler with tiled numbered units

3. In the second ruler of the sequence (Fig. 6), the units are tiled and delimited by marks. In the previous ruler, the *ruler with tiled units* (Fig. 5), marks are intentionally omitted to discourage counting them (CM strategy), allowing the emphasis to remain on the units (INT strategy).
4. As a solution to avoid counting units repeatedly during measurements, the third ruler in RbySteps introduces numbered units in a tiled arrangement (Fig. 7). This ruler again underscores the importance of measuring both aligned and displaced objects. When measured from the zero mark, the number signifies an accumulation of distance from that point. However, measuring from a different mark alters the interpretation, leading to the more common practice of counting the units constituting the length of the object (INT strategy).
5. The next step involves eliminating the units to create a ruler closely resembling the conventional version, resulting in the fourth ruler consisting solely of marks (Fig. 8). The crucial concept here lies in recognizing marks as unit delimiters rather than standalone units. Special attention is needed to understand the trace left by the removed unit, requiring the counting of these



Fig. 8 Ruler with marks



Fig. 9 Ruler with numbered space intervals

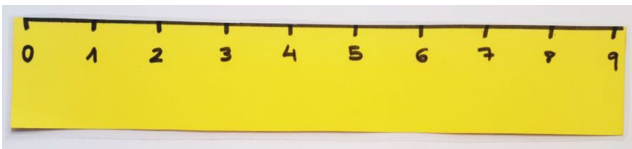


Fig. 10 Ruler with numbered marks

traces or spacing intervals between marks during measurements (INT strategy).

6. Similar to the third ruler of the set, the *ruler with tiled numbered units* (Fig. 7), the subsequent one (Fig. 9) introduces numbers indicating the accumulated space intervals (or units) from the zero mark to the corresponding location. However, in order for the units to remain visible, these numbers are positioned at the center of each space interval (or unit), deviating from the conventional ruler where numbers typically appear under each mark. This incorporation of numbers again emphasizes the need to concentrate on measurements from various points on the ruler, considering both aligned and displaced objects).
7. After comprehending the roles of numbers and marks on the measuring instrument, along with the display of units, the final ruler in RbySteps is presented (Fig. 10). On this ruler, numbers are positioned below the marks, mirroring the conventional ruler. This prompts the incorporation of the number zero on the initial mark, as, at this juncture, there is no accumulation of distance. The zero mark serves as the boundary to the left of the initial unit encompassing the ruler.

The concluding step in the sequence involves the use of a conventional ruler. Conceptually, this ruler closely resembles the last one proposed in the set—the *ruler with numbered marks*.

Intervention in the Control Group

In the control group, the intervention was aligned with traditional teaching methods of measuring length with a ruler. However, it did not gradually bridge the connection between measurement with discrete units and measurement that presents length as a continuous magnitude. Initially, a unit of measurement recognized by the subgroup was established, adhering to the concept of standardization for measurement communication proposed by Lehrer (2003). The participants longitudinally arranged units similar to those in Fig. 4 on strips of cardboard with integer measurements, ensuring no overlap or gaps between units—a consideration inspired by Lehrer’s tiling concept (2003). Following measurement with discrete units, a conceptually conventional ruler (Fig. 10) was introduced. Following the experimental group in the study by Levine et al. (2009), the results of measurements with this ruler were verified by placing units on top of the ruler along the strip of cardboard to help visualize the units.

Post-Test

One week after the interventions, the teachers administered the test of the “Evaluation instrument” section (post-test) to the individual pupils a second time.

Analysis

The data obtained were analyzed using the Statistical Package for the Social Sciences, SPSS, version 26. The number of correct answers was treated as a numerical variable and computed as a percentage.

As detailed in the section “Evaluation instrument: test”, the strategies that children used to measure were inferred from the results of each task. The following categories are established: EP strategy (reading the number coinciding with the end of the object or reading the number at the endpoint), CM strategy (counting marks), OTH (others), or correct response.

For example, in Fig. 1, if the student indicated “2” or “6” as the result, it was categorized as using the EP strategy; if the student indicated the result “5”, it was categorized as using the CM strategy; if the student indicated “4”, it was categorized as correct response. In the OTH category, for example, the student indicated “5” as the result in the task in Fig. 3, as the result does not correspond to any of the previous categories, EP or CM.

Given the nature of the data, non-parametric statistics are used. The association between the performance of the experimental and control groups for each of the test tasks was examined through χ^2 tests for independent samples).

Table 1 Percentage of correct measurements in the experimental and control group in the pre- and post-test, and significance of mean pre- and post- intervention differences

Task	Experimental group		Wilcoxon test	Control group		Wilcoxon test
	PRE	POST	For related samples	PRE	POST	For related samples
M1	21,1%	89,5%	$Z=-3.61, p<0.0$	0%	11,8%	$Z=-1.41, p=0.16$
M2	10,5%	89,5%	$Z=-3.87, p<0.00$	0%	5,9%	$Z=-1.00, p=0.32$
M3	10,5%	84,2%	$Z=-3.50, p<0.00$	0%	5,9%	$Z=-1.00, p=0.32$
M4	73,7%	94,7%	$Z=-2.00, p=0.04$	100%	100%	$Z=0, p=1.00$
M5	10,5%	89,5%	$Z=-3.87, p<0.00$	0%	5,9%	$Z=-1.00, p=0.32$
M6	5,3%	89,5%	$Z=-3.77, p<0.00$	0%	5,9%	$Z=-1.00, p=0.32$
M7	15,8%	89,5%	$Z=-3.50, p<0.00$	0%	5,9%	$Z=-1.00, p=0.32$
M8	78,9%	89,5%	$Z=-0.82, p=0.41$	100%	100%	$Z=0, p=1.00$

Table 2 Frequency and percentage of strategies used by both groups in the pre-test

	Correct	CM	EP	OTH
M1	4 (11.1%)	8 (22.2%)	24 (66.7%)	0
M2	2 (5.6%)	2 (5.6%)	32 (88.8%)	0
M3	2 (5.6%)	5 (13.9%)	29 (80.5%)	0
M4	31 (86.1%)	3 (8.3%)	1 (2.8%)	1 (2.8%)
M5	2 (5.6%)	6 (16.7%)	27 (74.9%)	1 (2.8%)
M6	1 (2.8%)	6 (16.7%)	27 (74.9%)	2 (5.6%)
M7	3 (8.3%)	6 (16.7%)	24 (66.7%)	3 (8.3%)
M8	32 (88.9%)	2 (5.6%)	1 (2.8%)	1 (2.8%)

Note: counting marks (CM); reading the number coinciding with the end of the object or reading the number at the endpoint (EP); others (OTH)

Comparisons between the measurements and strategies conducted in the pre- and post-study were carried out using the Wilcoxon test for related samples (Z).

Results

Correctness of Measurements

In the pre-test, both the experimental and control groups obtained percentages below 22% for correct measurements with the displaced object (M1, M2, M3, M5, M6, and M7): these percentages can be found in Table 1. For these measurements, no statistically significant differences were observed between the performances of the experimental and control groups ($\chi^2=1.89, p=0.17$ for M2, M3 and M5; $\chi^2=0.92, p=0.34$ for M6 and $\chi^2=2.93, p=0.09$ for M7), except for task M1 ($\chi^2=4.03, p=0.04$), where the experimental group obtained better results than the control group.

Continuing with the results obtained by the experimental and control groups in the pre-test, the tasks that positioned the object at the zero mark of the ruler (M4 and M8) achieved higher percentages of correct responses. Here, however, there were differences in the groups ($\chi^2=5.19, p=0.02$ for M4 and $\chi^2=4.02, p=0.04$ for M8), with the complete control group measuring correctly in these two tasks.

After the intervention with RbySteps, in the post-test, it can be observed that the percentages of correct responses significantly increased in the experimental group, being statistically confirmed for all measurements except for M8 (Table 1). On the other hand, in the control group, there were no statistically significant pre- or post- intervention differences for any of the measurements.

Measurement Strategies

Because there were no differences in the results of the pre-test measurements in virtually any of the tasks (except M1), the pre-test results of the experimental and control groups are presented together. Table 2 shows the percentage of correct answers and the strategies used in the case of incorrect answers (EP, CT, OTH). It worth noticing that, in measurements M4 and M8, using the EP strategy (reading the number at the end) implies indicating zero (far left) as the answer, as reading the number to the right of the object would give a correct answer.

As differences were observed in the experimental group after the intervention, it is delved into the use of strategies in this group. Table 3 displays the strategies employed in each measurement before and after the intervention. The last column of the table presents the mean difference statistic for related samples, as the strategies before and after the intervention are recorded for the same participant. Significant differences are observed in the measurements with the object displaced (M1, M2, M3, M5, M6, and M7), with no such differences appearing when the object was located at the zero mark of the ruler (M4 and M8).

Discussion

RbySteps Vs. Traditional Instruction in Length Measurement with a Ruler

One of the aims of this study was to evaluate the effectiveness of RbySteps, a didactic approach designed to enhance

Table 3 Frequency and percentage of strategies, and mean differences in the pre- and post-test for the experimental group

	PRE				POST				Wilcoxon test for related samples
	Correct	CM	EP	OTH	Correct	CM	EP	OTH	
M1	4 (21.1%)	5 (26.3%)	10 (52.7%)	0	17 (89.5%)	1 (5.3%)	1 (5.3%)	0	Z= -2.67; p=0.01
M2	2 (10.5%)	2 (10.5%)	15 (78.9%)	0	17 (89.5%)	1 (5.3%)	1 (5.3%)	0	Z= -3.70; p<0.00
M3	14 (73.7%)	2 (10.5%)	3 (15.8%)	0	16 (84.2%)	1 (5.3%)	2 (10.5%)	0	Z= -3.34; p<0.00
M4	14 (73.7%)	3 (15.8%)	1 (5.3%)	1 (5.3%)	18 (94.7%)	0	0	1 (5.3%)	Z=0; p=1
M5	2 (10.5%)	4 (21.1%)	12 (63.1%)	1 (5.3%)	17 (89.5%)	0	2 (10.5%)	0	Z= -2.02; p=0.04
M6	1 (5.3%)	3 (15.8%)	13 (68.4%)	2 (10.5%)	17 (89.5%)	1 (5.3%)	1 (5.3%)	0	Z= -3.09; p=0.02
M7	3 (15.8%)	3 (15.8%)	10 (52.7%)	3 (15.8%)	1 (5.3%)	1 (5.3%)	1 (5.3%)	0	Z= -3.39; p<0.00
M8	15 (78.9%)	2 (10.5%)	1 (5.3%)	1 (5.3%)	17 (89.5%)	0	0	2 (10.5%)	Z=0.11; p=0.91

Note: counting marks (CM); reading the number coinciding with the end of the object or reading the number at the endpoint (EP); others (OTH)

understanding of the role of measurement units on the conventional ruler when measuring lengths. This is achieved through a sequential set of six rulers that simulates the construction of a ruler, lending meaning to its elements—numbers, marks, and units.

In the pre-test, both the experimental and control groups yielded similar results. Coinciding with research proposing measurement tasks involving objects aligned and displaced with respect to the zero mark of the ruler (Congdon et al., 2018; Levine et al., 2009; Solomon et al., 2015), performance when the object was aligned at the zero mark was significantly higher than when the object was displaced. Specifically, Solomon et al.'s study (2015), conducted with children in early childhood education and 2nd grade of primary education, revealed that the early childhood education group did not exceed 25% correct answers in ruler measurement tasks with numbers and displaced objects in their experiments 1 and 2, mirroring our sample's performance in the pre-test in measurements M1, M2, M3, M5, M6, and M7. Levine et al.'s study (2009), involving 2nd graders, reported approximately 30% correct results in the pre-test measurements with the displaced object.

It appears that the children, despite lacking explicit instruction in the use of the ruler, are familiar with and employ the mechanical measurement procedure: placing the object at the zero mark and reading the number that aligns with the right-hand end (EP). According to Solomon et al. (2015) and Gómezescobar et al. (2023), in displaced object task, participants often fail to consider that numbers represent the accumulation of units up to that specific point. This is because they fail to consider the starting point for measurement. Therefore, the lack of understanding of the role of numbers in the measuring instrument hinders the comprehension of length measurement at these ages.

Strategies Employed in Length Measurement with a Ruler Before the Interventions

In the pre-test, in tasks involving the displaced object (M1, M2, M3, M5, M6, and M7), the predominant strategy was

to read the number coinciding with the end of the object or reading the number at the endpoint (EP), leading to incorrect results. Thus, in measurements with the object aligned at the zero mark (M4 and M8) where the percentage of correct answers was significantly higher, use of the EP strategy was minimal and would correspond to reading the number on the far left, representing zero.

The predominant use of the reading the number at the endpoint strategy (EP) in these measurements aligns with findings from several studies. Bragg and Outhred (2004), Kamii (1995), and Levine et al. (2009) also investigated tasks involving the displaced object, reporting varying percentages of EP strategy use, ranging from 20% in 2nd graders (Levine et al., 2009), and 5% in 4th graders (Kamii, 1995), to 24% in 6th graders (Bragg & Outhred, 2004). Gómezescobar et al. (2023), working with a similar age group as in the present study (the final year of early childhood education and 1st graders), found the EP strategy to be predominant, with percentages of 57.4% and 60.6%, respectively. Similarly, the results appendix of Gómezescobar's (2020) thesis demonstrates that over 50% of children aged 4 to 7 use the reading the number at the end strategy (EP) to measure with a ruler with numbers under the marks, similar to the ruler used in the evaluation instrument of this study.

Strategies Employed in Length Measurement with a Ruler after the Interventions

Following the didactic interventions, there was a noticeable shift in trend within the experimental group. This change was not observed in the control group, however. Let us recall that the control group adhered to traditional methods of teaching length measurement, initially utilizing discrete units for measurement before transitioning to ruler measurements. Additionally, to underscore the significance of units on the ruler, measurements taken with the ruler were verified and compared by placing measurement units atop the ruler along the cardboard strip. It is worth noticing that in all groups, measurements were considered both with the object aligned and displaced with respect to the

zero mark on the ruler. Similar guidelines were employed in experimental group in the study conducted by Levine et al. (2009). However, in contrast to the participants in their study, the children in our control group showed no significant improvement after this type of instruction. This lack of performance could be related to the age of the participants, 7–8 years in the group of Levine et al. (2009) and 5–6 years, in our case.

Conversely, the experimental group in our study experienced benefits stemming from the RbySteps instructional approach, involving the use of six tailored rulers that systematically guide the construction of a ruler. These advantages were manifested in an increase in the number of correct measurements involving displaced objects with respect to the zero mark of the ruler. It is worth noting that such tasks necessitate a more profound conceptual understanding, as they require interpretation of the role of measurement units and the starting point for measurements. The rise in correct measurements corresponds to a decline in error-inducing strategies. Specifically, there was a reduction in the use of the strategy where students read the number coinciding with the end of the object (EP) and the strategy where they count the marks on the ruler (CM).

Upon closer examination of the erroneous strategies employed for expressing measurements in the experimental group, it is evident that two children consistently used the CM and EP strategies, respectively. This observation suggests a lack of interpretation regarding the role of units on the ruler. Given the small number of participants involved (only two), it is challenging to ascertain the progression between strategies, as proposed by works such as those by Levine et al. (2009), Solomon et al. (2015), and Gómezescobar et al. (2023). The theoretical progression of strategies typically starts with reading the point at the end (EP), considered a less advanced strategy than mark counting (CM), which stands closer to action on the interval (INT). The most conceptually sophisticated strategy in terms of measurement involves acting on the spatial interval or unit, along with its subsequent counting (INT). This approach demonstrates a keen understanding of the unit within the measuring instrument. This evolution, first EP and then CT, could lead to children counting the units comprising the length of the object being measured (INT).

In this sense, it is worth noting that, in the overall analysis of the post-test of the experimental group, there is a decrease in the “Others” strategy (OTH). This suggests that the children were responding in accordance with the measurement strategies. Although these answers were not correct, they were inclined towards length measurement.

To conclude, alignment was found with previous studies emphasizing the importance of highlighting the meaning and interpretation of the unit within the measuring instrument

(the ruler) and discouraging the mere mechanical teaching of ruler usage. Specifically, implementing RbySteps—a proposal that sequences ruler construction through six ad-hoc rulers and provides measurements of objects aligned and displaced from the zero mark—yields effective conceptualization of measuring length with a ruler.

Hypothetical Learning Trajectory

Given the insights into how children learn measurement and the proposed didactic sequencing, RbySteps, in line with the second aim of this work, a Hypothetical Learning Trajectory (HLT) is proposed, which is exclusively focused on measuring with a ruler. This HLT could complement the learning trajectory (LT) presented by Clements et al. (2017), as their proposal is more general, encompassing length measurement in its entirety. The proposed HLT for length measurement with a ruler extends Clements et al.’s (2017) LT by providing a more detailed and structured progression of levels specifically focused on the use of rulers for length measurement. It introduces intermediate levels that address common misconceptions and strategies used by children, such as the counting marks strategy (CM) and the correct interpretation of numbers and marks on the ruler. By doing so, it offers a more granular approach to understanding and teaching length measurement, which can help educators better support students’ learning and address specific challenges they may encounter.

The HLT for length measurement with a ruler is outlined in Table 4 and is based on understanding ruler design through the gradual introduction of its elements: numbers, marks, and units. This foundational premise aligns with the approach taken in RbySteps, where each level of the developmental progression is determined by a specific ruler.

The first level of the HLT, “Units measurer”, corresponds, concerning the ruler, to the “End-to-End Length Measurer” level proposed by Clements et al. (2017). This level specifies that students “lay units end to end to create measures for comparison” and “use rulers with minimal guidance and some success”, indicating that children may “ignore the zero point or report the number at the far end of the ruler” (p.63). This aligns with the pre-test results, showing a preference for using numbers on rulers following the EP strategy.

The next five levels of the proposed HLT would align with the “Length Unit Relater and Repeater” level of Clements et al.’s (2017) LT. While this level focuses on the iteration of units of measurement, it constitutes a more global model for length measurement. Regarding ruler usage, it emphasizes doing so “with minimal guidance and more success than at the previous level. Measures a length accurately with a ruler if aligned to the zero point”. Additionally, it notes the misconception that “when an object does not align

Table 4 HLT for length measurement with a ruler

Developmental Progression	Actions on objects	Representative task
Unit measurer	Tiles units along an object with no gaps or overlaps, counts units to state the measurement	To measure objects with units similar to those on RbySteps' rulers (Fig. 4)
Units tiled on a support measurer	Understands that embedding units in a holder avoids having to tile units each time they want to measure. Measures objects aligned at the zero mark and offset with a ruler showing tiled units.	To measure objects with the ruler with tiled units (Fig. 5)
Units tiled on a support delimited by a mark measurer	Understands marks as delimiters of units, not as units in themselves. Measures objects aligned at the zero mark and displaced with a ruler with units tiled and delimited by marks.	To measure objects with the ruler with tiled units delimited by marks (Fig. 6)
Numbered units tiled on a support measurer	Measures objects aligned at the zero mark and displaced with a ruler with tiled numbered units. Understands the significance of numbers as indicators of distance accumulation from the zero mark to the number location.	To measure objects with the ruler with tiled numbered units (Fig. 7)
Ruler with a mark measurer	Measures aligned and displaced objects with a ruler with marks. Understands marks as delimiters of units, not as units in themselves. Understands units as the interval of space or trace left by units that have been removed from the previous ruler.	To measure objects with the ruler with marks (Fig. 8)
Ruler with numbered space intervals measurer	Measures aligned and displaced objects with a ruler with marks and numbers in the space intervals between marks. Understands units as the interval of space or trace left by units that have been removed from the previous ruler and does not attribute the entity of unit to the marks. Understands the significance of numbers as indicators of the accumulation of distance from the zero mark to the location of the number.	To measure objects with the ruler with numbered space intervals (Fig. 9)
Ruler with numbered marks measurer	Measures aligned and displaced objects with a ruler with marks and numbers under them. Understands units as the interval of space between two marks and does not attribute unit meaning to the marks. Understands the meaning of numerals as indicators of the accumulation of distance from the zero mark to the number location.	To measure objects with the ruler with numbered marks (Fig. 10)
Standard ruler measurer	Measures aligned and displaced objects with a standard ruler. Understands units as the interval of space between two marks and does not attribute unit meaning to the marks. Understands the meaning of numerals as indicators of the accumulation of distance from the zero mark to the location of the numeral.	To measure objects with a standard ruler

to start at zero on a ruler, children may include the initial tick mark when they count tick marks to find length" (p. 64), which corresponds to the use of the counting marks strategy (CM), considered more advanced in terms of length conceptualization than reading the number at the end of the object (EP) (Gómezescobar et al., 2023).

The primary aim of the abovementioned five levels is to facilitate the transition or visualization of units on the measuring instrument. Specifically, addressing the misuse of the CM strategy is the focus of the third level, called "Units tiled on a support delimited by marks measurer," which introduces marks on the measuring instrument. The meaning of the numbers on the ruler is first introduced in the fourth level, "Numbered units tiled on a support measurer." The subsequent levels, review and combine interpretations of numbers, marks, and space intervals as different elements, ultimately leading to the correct use of the standard ruler.

The last two levels of the proposed HLT, "Ruler with numbered marks measurer" and "Standard ruler measurer,"

would fit within the "Consistent Length Measurer" level (Clements et al., 2017, p. 65). At this level, specific aspects related to using the ruler and understanding its design are delineated, including the zero point of the ruler, the accumulation of distance, and the broken ruler task, which conceptually resembles the shifted object task. While these aspects are introduced in previous levels, the design of the rulers used in these two levels is similar or identical to the conventional ruler intended for use at the level proposed by Clements et al. (2017).

Limitations and Future Directions

One of the most notable limitations of this study lies in the sample being convenience-based, which may restrict the generalizability of the findings. It would be beneficial to increase both the size and diversity of the sample, in terms of both quantity and geographical regions, to enhance the applicability of the results.

Another limitation of the study is related to inferring the strategies used by the participants based on numerical results of their measurements. Although this technique is deployed by other reference studies (Bragg & Outhred, 2004; Cogndon et al. 2018; Cullen & Barrett, 2010; Irwin et al., 2004; Kamii, 1995; Levine et al., 2009; Solomon et al., 2015), children verbalizing strategies through interviews is suggested as a prospective approach. In addition, this would allow for further exploration of the possible progression of strategies: reading the endpoint (EP)-counting marks (CM)-action on the interval (INT).

Another limitation may be related to the use of manipulative material during interventions and graph-based assessment. The transfer between different representations could affect no children demonstrate their knowledge. Therefore, in addition to the aforementioned interviews, including, during instruction, worksheets with measurement situations that require representation by rulers and cardboard strips is suggested.

In addition, it is proposed to explore the Hypothetical Learning Trajectory (HLT) focused on measuring lengths with rulers. To validate its relevance, it is essential to carry out a detailed analysis of each level of the HLT and the activities associated with them. In addition, it is common to make adjustments to HLTs to improve their effectiveness. This approach allows the HLT to be adapted according to the specific needs of the group of learners and to maximize the learning outcomes of this fundamental skill.

Conclusion

In summary, this study reinforces the consensus found in earlier research, underlining the crucial significance of emphasizing the significance and interpretation of the unit on the measuring instrument, such as the ruler. The findings strongly advocate against a purely mechanical approach to teaching ruler usage. Notably, the implementation of RbySteps, a pedagogical framework that systematically sequences ruler construction via six purpose-designed rulers and offers measurement scenarios involving objects both aligned and displaced from the zero mark, demonstrates a successful proposal for fostering a profound conceptual understanding of measuring length with a ruler. Moreover, this approach gives rise to the formulation of a Hypothetical Learning Trajectory (HLT) for length measurement, with each level corresponding to a step in the RbySteps sequence.

Based on the findings that children often fail to consider the starting point for measurement and the role of units in the measuring instrument, future research should examine instructional methods that explicitly address these conceptual gaps. Based on the findings that the RbySteps approach

significantly improves understanding of measurement units and starting points, designers of early childhood learning should consider integrating sequential ruler construction activities into their curricula to enhance conceptual understanding of length measurement.

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Declarations

Ethical Approval The method presented has obtained a favourable report from the ethics committee of the Universidad Autónoma de Madrid (CEI-121- 2444).

Conflicts of Interest I have no conflicts of interest to disclose.

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