

Short Communication

Pythagorean-hodograph curves of Tschirnhaus type are sinusoidal spirals [☆]

Javier Sánchez-Reyes

Universidad de Castilla-La Mancha, IMACI, Department of Applied Mechanics, ETS Ingeniería Industrial Ciudad Real, 13071 Ciudad Real, Spain

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ABSTRACT

Recently, Bizzarri et al. (2021) discussed the so-called Pythagorean-hodograph curves of Tschirnhaus type, a generalization to higher degrees of Tschirnhausen cubic. We recall that these curves in Bézier form coincide with the *typical curves* introduced by Mineur et al. (1998), as well as with a classical family of sinusoidal spirals. Therefore, they all enjoy the same properties, such as the rational character of their offsets or the existence of only one curve (up to similarities) for each degree. By elucidating this connection among curves of Tschirnhaus type, typical curves, and sinusoidal spirals, we rederive several relevant results found by Bizzarri et al. (2021).

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1. Introduction: Pythagorean-hodograph curves of Tschirnhaus type

In a recent Short Communication in this journal, Bizzarri et al. (2021) put forward a generalization to general degree n of the celebrated Tschirnhaus' (or Tschirnhausen) cubic, aka *l'Hôpital's cubic* or *Trisectrix of Catalan*. The Bézier polygon $\{\mathbf{b}_k\}_{k=0}^n$ of these planar curves $\mathbf{b}(u)$ is constructed starting from an initial control leg $\Delta\mathbf{b}_0 = \mathbf{b}_1 - \mathbf{b}_0$ of length $L \neq 0$. Then, by successively multiplication of $\Delta\mathbf{b}_0$ by a constant matrix \mathbf{M} expressing uniform dilation $\lambda > 0$ plus rotation of angle $\varphi \neq 0$, we generate the control legs $\Delta\mathbf{b}_k = \mathbf{b}_{k+1} - \mathbf{b}_k$:

$$\Delta\mathbf{b}_k = \mathbf{M}^k \Delta\mathbf{b}_0, \quad k = 0, \dots, n-1, \quad \mathbf{M} = \lambda \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}. \quad (1)$$

Equivalently, using complex products, the control legs form a geometric progression:

$$\Delta\mathbf{b}_k = \mathbf{z}^k \Delta\mathbf{b}_0, \quad k = 0, \dots, n-1, \quad \mathbf{z} = \lambda e^{i\varphi}. \quad (2)$$

We recall that these *curves of Tschirnhaus type* coincide with the *typical curves* in (Mineur et al., 1998), and also with a family of offset-rational sinusoidal spirals identified by Ueda (1997). Consequently, several remarkable properties listed in Bizzarri et al. (2021) can be rederived in a simpler way from these previous works, or come as direct consequences of this connection. These properties include the existence of only one curve of Tschirnhaus type for each n , the rational character of their offsets, a simple condition in terms of (φ, λ) to guarantee monotone curvature, and a straightforward formula for the rotation index as a function of φ .

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E-mail address: Javier.SanchezReyes@uclm.es.

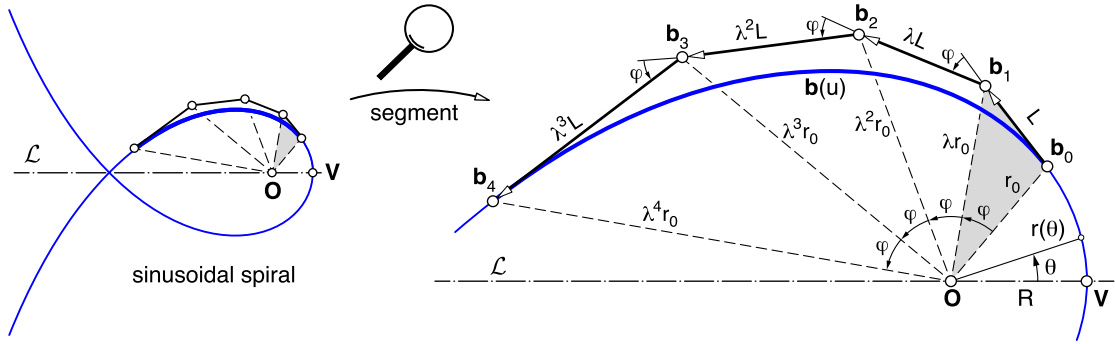


Fig. 1. Geometric constraints (φ, λ) characterizing a Bézier curve $\mathbf{b}(u)$ of Tschirnhaus type, i.e., a sinusoidal spiral segment (degree $n = 4$).

2. Coincidence with typical curves and sinusoidal spirals

Spelling out condition (2), the control polygon is characterized by two geometric constraints (Fig. 1):

- (i) Constant supplementary angle φ between consecutive control legs.
- (ii) Constant ratio λ between their lengths.

These two constraints define what Mineur et al. (1998) dubbed *typical curves*. Therefore, as Farin (2006) remarks, they coincide with the curves in matrix form (1). Such constrained curves are based on earlier research (Higashi et al., 1988; Higashi, 1993) on the cubic case. Mineur (2003) proposed them for styling surface modeling because we can easily ensure the strict monotonicity of their curvature with a proper choice of (φ, λ) :

$$\cos \varphi > \begin{cases} 1/\lambda \rightarrow \text{increasing} \\ \lambda \rightarrow \text{decreasing} \end{cases} \text{curvature.} \quad (3)$$

To identify these curves as sinusoidal spirals, we rewrite characterization (2) using a suitable origin of coordinates \mathbf{O} . Given the initial control leg $\Delta \mathbf{b}_0$, consider the point \mathbf{O} (Fig. 1) that sees it with an angle φ , and such that the ratio of distances from \mathbf{O} to $\mathbf{b}_0, \mathbf{b}_1$ is $r_1/r_0 = \lambda$. Using complex arithmetics:

$$\mathbf{b}_1 - \mathbf{O} = (\mathbf{b}_0 - \mathbf{O})\mathbf{z} \rightarrow \mathbf{O} = \frac{\mathbf{b}_1 - \mathbf{b}_0\mathbf{z}}{1 - \mathbf{z}}, \quad \mathbf{z} = \lambda e^{i\varphi}. \quad (4)$$

By rotation φ around \mathbf{O} and dilation λ , the triangle $\Delta \mathbf{O}\mathbf{b}_0\mathbf{b}_1$ (shaded in Fig. 1) transforms to its similar $\Delta \mathbf{O}\mathbf{b}_1\mathbf{b}_2$, whose edge $\Delta \mathbf{b}_1$ hence fulfills characterization (2) for $k = 1$. Repeating this process yields all adjacent triangles $\Delta \mathbf{O}\mathbf{b}_k\mathbf{b}_{k+1}$, also similar, with control legs $\Delta \mathbf{b}_k$ (2) for each k . Therefore, in a coordinate system with center \mathbf{O} (4), the control points \mathbf{b}_k also form a geometric progression with common ratio \mathbf{z} :

$$\mathbf{b}_k - \mathbf{O} = (\mathbf{b}_0 - \mathbf{O})\mathbf{z}^k, \quad k = 0, \dots, n. \quad (5)$$

In terms of the polar coordinates (r_k, θ_k) of \mathbf{b}_k , this equivalent characterization translates to

- (I) Constant incremental polar angle $\varphi = \theta_{k+1} - \theta_k$ between consecutive control points $\mathbf{b}_k, \mathbf{b}_{k+1}$.
- (II) Constant ratio $\lambda = r_{k+1}/r_k$ between their polar radii, so that the points \mathbf{b}_k lie on a *logarithmic spiral* of center \mathbf{O} .

Sánchez-Reyes (2007) observed that the constraints (I), (II) define a certain family of *sinusoidal spirals* (Gomes Teixeira, 1909; Lawrence, 1972) with pole at \mathbf{O} , described in the classical book (Loria, 1911) and obtained by raising a line to the n th power in the complex plane. If the reference angle $\theta = 0$ corresponds to the symmetry axis \mathcal{L} of the spiral, passing through and its pole \mathbf{O} and vertex \mathbf{V} , the polar radius of the spiral is given by

$$r(\theta) = R \cos^{-n}(\theta/n), \quad R = r(0) = \|\mathbf{O}\mathbf{V}\|, \quad (6)$$

where R denotes a scale factor. Remarkably, this subset of Bézier sinusoidal spirals is closed with respect to differentiation and integration, a property reflected in the equivalence of characterizations (2) and (5). Indeed, differentiating a degree- n sinusoidal spiral (5) in Bézier form $\mathbf{b}(u)$ yields as hodograph $\mathbf{b}'(u)$ another sinusoidal spiral, of degree $n - 1$ and centered at $\mathbf{O} = \mathbf{0}$. Therefore, by (5) the Bézier points $\mathbf{b}'_k = n\Delta \mathbf{b}_k$ of $\mathbf{b}'(u)$ must form a geometric progression, which is precisely characterization (2) for $\mathbf{b}(u)$.

Ueda (1997, 2001) first reported on these spirals in the CAGD field in Bézier form, via a pedal-point construction. Later, Sánchez-Reyes (2002, 2007) identified these spirals as also belonging to the special subset of *p-Bézier curves*, i.e., Bézier curves in polar coordinates (Sánchez-Reyes, 1990), and noted that they coincide with Mineur et al.'s typical curves.

3. Properties of sinusoidal spirals

In Section 2, we concluded that curves of Tschirnhaus type coincide with typical curves, which in turn are segments of certain sinusoidal spirals. There exist two ways to express them: either in a CAGD fashion, as Bézier segments (2), or in a classical way (6) in polar coordinates. Since these curves enjoy the same properties, some listed in (Bizzarri et al., 2021) come as consequences of this connection or can be rederived in a simpler manner:

- Unicity for a given degree n up to similarities (Theorem 4.1): Just observe that, in definition (6), $r(\theta)$ only depends on n and the scale factor R .
- Rational offsets (Lemma 2.1 and Corollary 4.4): Ueda (1997) notes that, for n odd, these sinusoidal spirals are Pythagorean-hodograph (Farouki, 2008); for n even, they have rational speed after rational quadratic reparameterization, that is, adopting an improper parameterization. Hormann and Zheng (2020) also noted this property of the constrained curves (2).
- Condition for monotone curvature (Lemma 3.1): It coincides with the inequalities (3) derived by Mineur et al. (1998). Geometrically, $\mathbf{b}(u)$ must not contain the vertex \mathbf{V} (Sánchez-Reyes, 2007).
- Rotation index: The normal vector $\mathbf{n}(\theta) = \exp(i(n-1)\theta/n)$ to the sinusoidal spiral (6) turns a net angle $\Delta = (n-1)\varphi$ on traversing $\mathbf{b}(u)$. The *rotation index* \mathcal{R} of $\mathbf{b}(u)$, i.e., the fraction of a complete revolution that the curve tangent (or \mathbf{n}) executes, is hence computable without integration and coincides with that of the control polygon:

$$\mathcal{R} = \frac{\Delta}{2\pi} = \frac{(n-1)\varphi}{2\pi}.$$

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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