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On the asymptotic behaviour of a variable exponent power law magnetostatic problem

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ABSTRACT

We introduce a variational scheme to study the asymptotic behaviour of magnetostatic problems, with a $p_h(x)$ -curl-type equation coming from the stationary Maxwell equations in magnetic media coupled with a nonstandard magnetic constitutive power law, which enables the explicit computation of the effective coefficients.

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1. Introduction

In different fields of science, it is fundamental to give a macroscopic description of heterogeneous media with a periodic microstructure whose period is smaller than the size of the sample we are interested in. The mathematical theory of homogenization focuses on the asymptotic analysis, as the period goes to zero, of boundary value problems which describe microscopically the behaviour of such heterogeneous media. See [1–7]. In this work, we are interested in the asymptotic analysis of a boundary value problem which turns out to be a formulation of the stationary Maxwell equations describing the microscopic behaviour of a magnetic field in a periodic non-linear heterogeneous medium with a nonstandard constitutive power law. The stationary Maxwell equations in magnetism are two fundamental laws governing the behaviour of magnetic fields interacting with media. Depending on the properties of the media, we may supplement the Maxwell equations by different constitutive relations as well as different boundary conditions. See for instance [8–13].

In the previous work [14], we have studied the asymptotic behaviour of stationary Maxwell equations with a standard linear constitutive magnetic relation in heterogeneous anisotropic composite materials with a periodic microstructure. So, in this work, we go further and consider a constitutive relation characterized by a variable exponent power law.

The main feature in this work is to study the homogenization of stationary Maxwell equations coupled with a two-phase variable exponent power law from a variational point of view through the study of the Γ -convergence of the associated sequence of energies. Γ -convergence is a variational convergence for sequences of integral functionals, defined in appropriated function spaces, useful to study the asymptotic behaviour of sequences of minimizers. See for instance [15–18]. We focus on the Γ -convergence of sequences of energies with two-phase variable exponents and we explicitly characterize the density of the Γ -limit functional. Γ -convergence of functionals with continuous variable exponents was previously studied in [19,20].

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We summarize the structure of this work. In Section 2, we state the problem and the main result. In Section 3, we recall the definition of variable exponent Lebesgue spaces as well as their main properties which will be used in Section 5 to prove the main result. Section 4 is dedicated to the so-called div-Young measures, particularly to such associated with two-phase sequences, which turn out to be the main tool used to study the Γ -convergence of the sequence of energies. In Section 5, first we recall the definition of Γ -convergence, and then the main theorem is proved. Section 6 illustrates applications of our result to the homogenization of two-phase variable exponent power laws.

2. Statement of the problem and main result

We consider a sample Ω of a three-dimensional material with a periodic structure of relative size $1/h$ formed by two alternate layers, one of material I and other of material II , with relative thickness α/h and $(1 - \alpha)/h$, respectively, normal to the direction of a given unit vector e . Materials I and II are assumed to be non-linear homogeneous materials with different positive magnetic permeabilities μ_I and μ_{II} , respectively. Thus, the magnetic permeability of the heterogeneous composite may be defined by the positive function μ_h given by

$$\mu_h(x) = \mu_I \chi_{(0,\alpha)}(hx \cdot e) + \mu_{II} (1 - \chi_{(0,\alpha)}(hx \cdot e)) \quad (1)$$

in Ω , where $\chi_{(0,\alpha)}(y)$ is the characteristic function of the interval $(0, \alpha) \subset (0, 1)$ extended by periodicity to the whole space $\mathbb{R}$. The magnetic properties of this composite at microscopic scale may be modelled by the stationary Maxwell equations

$$\begin{cases} \operatorname{curl} H = J \\ \operatorname{div} B = 0 \end{cases} \quad (2)$$

in Ω , where H and B represent the magnetic field and the magnetic induction, respectively, for a given total current density J . If the composite has a perfectly conducting boundary $\partial\Omega$, we shall assume that

$$B \cdot n = 0 \quad \text{and} \quad H \times n = J_s \quad (3)$$

on $\partial\Omega$, where n stands for the outward normal to the boundary, and J_s for the surface current. The main feature in this work is to assume that the composite is characterized by the constitutive relation

$$H = \mu_h^{-1} |B|^{p_h-2} B, \quad (4)$$

where $\mu_h^{-1} = \frac{1}{\mu_h}$ and the variable exponent $p_h : \Omega \rightarrow [2, +\infty)$ oscillates between p and q depending on the laminate microstructure of Ω , i.e.

$$p_h(x) = p \chi_{(0,\alpha)}(hx \cdot e) + q (1 - \chi_{(0,\alpha)}(hx \cdot e)), \quad (5)$$

with $2 \leq p \leq q < +\infty$. Moreover, we assume that the current density J is the curl of a given vector potential j in Ω .

If we substitute the constitutive relation (4) into the stationary Maxwell equations (2) coupled with the boundary conditions (3), then the magnetic induction B satisfies the boundary value problem

$$\begin{cases} \operatorname{curl} \left(\mu_h^{-1}(x) |B(x)|^{p_h(x)-2} B(x) \right) = \operatorname{curl} j(x) & \text{in } \Omega \\ \operatorname{div} B = 0 & \text{in } \Omega \\ B \cdot n = 0 & \text{on } \partial\Omega \\ \mu_h^{-1} |B|^{p_h-2} B \times n = j \times n & \text{on } \partial\Omega. \end{cases} \quad (6)$$

Let us assume that Ω is a simply connected bounded domain with a Lipschitz continuous boundary $\partial\Omega$. Since the magnetic induction B is divergence-free in Ω and $B \cdot n = 0$ on $\partial\Omega$, then there exists a unique vector potential u such that

$$B = \operatorname{curl} u \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u \cdot n = 0 \quad \text{on } \partial\Omega,$$

see [13,21–23]. Thus, the magnetostatic problem (6) may be rewritten using such vector potential formulation so that

$$\begin{cases} \operatorname{curl} \left(\mu_h^{-1}(x) |\operatorname{curl} u(x)|^{p_h(x)-2} \operatorname{curl} u(x) \right) = \operatorname{curl} j(x) & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega \\ \mu_h^{-1} |\operatorname{curl} u|^{p_h-2} \operatorname{curl} u \times n = j \times n & \text{on } \partial\Omega. \end{cases} \quad (7)$$

In this way, the magnetic properties of the composite material at microscopic scale may be deduced from the previous boundary value problem for the vector potential u in Ω . Now, the question is: How does the composite behave at macroscopic scale? That is, when the relative size h goes to $+\infty$, does the composite behave like a homogeneous material? In order to give an answer to these questions, we intend to study the asymptotic behaviour, as the parameter h goes to $+\infty$, of the sequence of solutions u_h of the family of boundary value problems (7) from a variational point of view. Our aim is to study the homogenization of $p_h(x)$ -curl systems of type (7) through the Γ -convergence of the family of associated energies. The main techniques used here were firstly applied in [14] to study the homogenization of magnetostatic problems for composite materials with any periodic microstructure, not necessarily a laminate one, whose constitutive relation was linear of type $H = \mu_h^{-1} B$. See also [24,25].

Before advancing in the main issue of this work, let us comment on the existence and uniqueness of weak solutions of problem (7) when the exponent p_h is constant equal to $p \geq 2$. If we assume that $\Omega \subset \mathbb{R}^3$ is a simply connected bounded domain with a Lipschitz continuous boundary $\partial\Omega$, the space $X^p(\Omega)^3 = \{u \in L^p(\Omega)^3 : \operatorname{curl} u \in L^p(\Omega)^3, \operatorname{div} u = 0 \text{ in } \Omega, u \cdot n = 0 \text{ on } \partial\Omega\}$ is a closed subspace of the Sobolev space $W^{1,p}(\Omega)^3$. Moreover, $X^p(\Omega)^3$ is a separable reflexive Banach space endowed with the norm $\|u\|_{X^p(\Omega)^3} = \|u\|_{L^p(\Omega)^3} + \|\operatorname{curl} u\|_{L^p(\Omega)^3}$. Thus, the weak formulation of problem (7) in the particular case $p_h = p$ reads as: given j , with $\operatorname{curl} j \in L^{p'}(\Omega)^3$, find $u_h \in X^p(\Omega)^3$ such that

$$\int_{\Omega} \mu_h^{-1}(x) |\operatorname{curl} u_h(x)|^{p-2} \operatorname{curl} u_h(x) \cdot \operatorname{curl} \varphi(x) \, dx = \int_{\Omega} j(x) \cdot \operatorname{curl} \varphi(x) \, dx$$

for every $\varphi \in X^p(\Omega)^3$. There is a unique solution u_h in $X^p(\Omega)^3$ of the weak formulation, see [26, Proposition 2.1]. Now, if p_h is a bounded function in Ω such that $2 \leq p_- \leq p_h(x) \leq p_+ < \infty$ for a.e. $x \in \Omega$, the weak formulation of the magnetostatic problem (7) reads as: given j , with $\operatorname{curl} j \in L^{p'_+}(\Omega)^3$, find $u_h \in X^{p_h(\cdot)}(\Omega)^3$ such that

$$\int_{\Omega} \mu_h^{-1}(x) |\operatorname{curl} u_h(x)|^{p_h(x)-2} \operatorname{curl} u_h(x) \cdot \operatorname{curl} \varphi(x) \, dx = \int_{\Omega} j(x) \cdot \operatorname{curl} \varphi(x) \, dx$$

for every $\varphi \in X^{p_h(\cdot)}(\Omega)^3$, where the space $X^{p_h(\cdot)}(\Omega)^3$ is defined by

$$X^{p_h(\cdot)}(\Omega)^3 = \left\{ u \in L^{p_h(\cdot)}(\Omega)^3 : \operatorname{curl} u \in L^{p_h(\cdot)}(\Omega)^3, \operatorname{div} u = 0 \text{ in } \Omega, u \cdot n = 0 \text{ on } \partial\Omega \right\}.$$

On the other hand, the magnetostatic problem (7) turns out to be the first-order optimality condition associated with the integral functional E_h defined by

$$E_h(u) = \int_{\Omega} \left(\frac{\mu_h^{-1}(x)}{p_h(x)} |\operatorname{curl} u(x)|^{p_h(x)} - j(x) \cdot \operatorname{curl} u(x) \right) dx \quad (8)$$

if $u \in X^{p_h(\cdot)}(\Omega)^3$, see Proposition 6.1. That is, the unique weak solution u_h of the boundary value problem (7) is the unique minimizer of the associated energy E_h . Therefore, the study of the asymptotic behaviour, as h goes to $+\infty$, of the sequence of solutions u_h of the family of systems (7) reduces to the study of the asymptotic behaviour of the sequence of minimizers u_h of the family of energies (8).

The functional E_h is a particular case of a general class of variable exponent power law functionals $F_h : L^{p_h(\cdot)}(\Omega)^3 \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$F_h(u) = \begin{cases} \int_{\Omega} f(p_h(x), \operatorname{curl} u(x)) dx & \text{if } u \in X^{p_h(\cdot)}(\Omega)^3 \\ +\infty & \text{otherwise,} \end{cases} \quad (9)$$

where the function $f : [2, +\infty) \times \mathbb{R}^3 \rightarrow [0, +\infty)$ is continuous on both variables, convex on the second one and there exist constants $c_1 > c_0 > 0$ such that

$$c_0 |\rho|^\lambda \leq f(\lambda, \rho) \leq c_1 (1 + |\rho|^\lambda) \quad (10)$$

for every $(\lambda, \rho) \in [2, +\infty) \times \mathbb{R}^3$.

We are interested in studying the Γ -convergence of this family of functionals as well as giving an explicit representation of the Γ -limit density through a three-dimensional minimization problem. In this way, our main result is the following one.

Theorem 2.1: *The sequence of functionals F_h defined in (9) under the nonstandard growth condition (10) is Γ -convergent, with respect to the strong topology of $L^p(\Omega)^3$, to the functional $F : L_{hom}(\Omega)^3 \rightarrow \mathbb{R} \cup \{\infty\}$ defined by*

$$F(u) = \begin{cases} \int_{\Omega} f_{hom}(\operatorname{curl} u(x)) dx & \text{if } \operatorname{curl} u \in L_{hom}(\Omega)^3 \\ +\infty & \text{otherwise.} \end{cases} \quad (11)$$

The density $f_{hom} : \mathbb{R}^3 \rightarrow [0, +\infty)$ is given by

$$f_{hom}(\rho) = \min_{\Lambda_1, \Lambda_2 \in \mathbb{R}^3} \left\{ \alpha f(p, \Lambda_1) + (1 - \alpha) f(q, \Lambda_2) : \begin{array}{l} \rho = \alpha \Lambda_1 + (1 - \alpha) \Lambda_2 \\ (\Lambda_1 - \Lambda_2) \cdot e = 0 \end{array} \right\}.$$

We define the space $L_{hom}(\Omega)^3$ as the space of all divergence-free functions u such that the functional $\int_{\Omega} f_{hom}(\operatorname{curl} u(x)) dx$ is finite. Such space turns out to be a subset of the intermediate space

$$L^p(\Omega)^3 + L^q(\Omega)^3 = \{w = v_1 + v_2 : v_1 \in L^p(\Omega)^3, v_2 \in L^q(\Omega)^3\}.$$

In the particular case of the sequence of power law energies E_h , we obtain the following Γ -convergence result.

Corollary 2.2: *The sequence of energies $\{E_h\}$ defined in (8) Γ -converges, with respect to the strong convergence of $L^p(\Omega)^3$, to the energy $E : L_{hom}(\Omega)^3 \rightarrow \mathbb{R} \cup \{\infty\}$ given by*

$$E(u) = \begin{cases} \int_{\Omega} (\psi(\operatorname{curl} u(x)) - j(x) \cdot \operatorname{curl} u(x)) \, dx & \text{if } \operatorname{curl} u \in L_{hom}(\Omega)^3 \\ +\infty & \text{otherwise,} \end{cases}$$

where $\psi : \mathbb{R}^3 \rightarrow [0, +\infty)$ is defined by

$$\psi(\rho) = \min_{\Lambda_1, \Lambda_2 \in \mathbb{R}^3} \left\{ \alpha \frac{\mu_I^{-1}}{p} |\Lambda_1|^p + (1 - \alpha) \frac{\mu_{II}^{-1}}{q} |\Lambda_2|^q : \begin{array}{l} \rho = \alpha \Lambda_1 + (1 - \alpha) \Lambda_2 \\ (\Lambda_1 - \Lambda_2) \cdot e = 0 \end{array} \right\}.$$

3. Variable exponent Lebesgue spaces. Intermediate spaces

In this work, we are interested in a special class of functionals, namely functionals of type (9) defined in Lebesgue spaces with variable exponents $p_h(x)$ which depend both on $h \in \mathbb{N}$ and on $x \in \Omega$. The Lebesgue spaces with variable exponents were introduced by Orlicz in [27], and later studied by Kováčik & Rábosník in [28], and Fan & Zhao in [29] (see also [30,32]). They play an important role in the study of the existence of solutions of non-linear minimization problems.

Let Ω be a bounded smooth domain in $\mathbb{R}^3$, and p_h be a function in $L^\infty(\Omega)$ such that $2 \leq p_h^- \leq p_h(x) \leq p_h^+ < +\infty$ for a.e. $x \in \Omega$, with $p_h^- = \inf_{x \in \Omega} p_h(x)$ and $p_h^+ = \sup_{x \in \Omega} p_h(x)$. The variable exponent Lebesgue space $L^{p_h(\cdot)}(\Omega)$ is defined by

$$L^{p_h(\cdot)}(\Omega) = \left\{ u \in L^1(\Omega) : \int_{\Omega} |\eta u(x)|^{p_h(x)} \, dx < \infty, \text{ for some } \eta > 0 \right\},$$

and the Luxemburg norm $\|\cdot\|_{L^{p_h(\cdot)}(\Omega)}$ defined by

$$\|u\|_{L^{p_h(\cdot)}(\Omega)} = \inf \left\{ \eta > 0 : \int_{\Omega} \left| \frac{u(x)}{\eta} \right|^{p_h(x)} \, dx \leq 1 \right\}$$

is a norm in $L^{p_h(\cdot)}(\Omega)$. The space $L^{p_h(\cdot)}(\Omega)$ equipped with the norm $\|\cdot\|_{L^{p_h(\cdot)}(\Omega)}$ is a separable and reflexive Banach space, see [28,29,32]. The variable exponent Lebesgue space $L^{p_h(\cdot)}(\Omega)^3$ is defined by

$$L^{p_h(\cdot)}(\Omega)^3 = \left\{ u = (u_1, u_2, u_3) : u_i \in L^{p_h(\cdot)}(\Omega) \text{ for every } 1 \leq i \leq 3 \right\},$$

and equipped with the norm

$$\|u\|_{L^{p_h(\cdot)}(\Omega)^3} = \sum_{i=1}^3 \|u_i\|_{L^{p_h(\cdot)}(\Omega)}.$$

If $p'_h \in L^\infty(\Omega)$ stands for the conjugate exponent of p_h , i.e. $\frac{1}{p_h(x)} + \frac{1}{p'_h(x)} = 1$ for a.e. $x \in \Omega$, then the variable exponent Lebesgue space $L^{p'_h(\cdot)}(\Omega)^3$ is the dual space of $L^{p_h(\cdot)}(\Omega)^3$, that is for every $v \in L^{p'_h(\cdot)}(\Omega)^3$ the functional $G : L^{p_h(\cdot)}(\Omega)^3 \rightarrow \mathbb{R}$ given by $G(u) = \int_{\Omega} u(x) \cdot v(x) \, dx$ is linear and continuous. Moreover, the Hölder inequality

$$\int_{\Omega} |u(x)| |v(x)| \, dx \leq \left(1 + \frac{1}{p_h^-} - \frac{1}{p_h^+} \right) \|u\|_{L^{p_h(\cdot)}(\Omega)^3} \|v\|_{L^{p'_h(\cdot)}(\Omega)^3}$$

holds for every $u \in L^{p_h(\cdot)}(\Omega)^3$ and $v \in L^{p'_h(\cdot)}(\Omega)^3$.

Let $\mathcal{D}(\Omega)^3$ be the space of all functions in $C^\infty(\Omega)^3$ with a compact support in Ω , $\mathcal{D}'(\Omega)^3$ be its dual and $\langle \cdot, \cdot \rangle$ be the duality between both. The space $\mathcal{D}(\Omega)^3$ is dense in $(L^{p_h(\cdot)}(\Omega))^3$, $\|\cdot\|_{L^{p_h(\cdot)}(\Omega)^3}$. For every $u \in L^{p_h(\cdot)}(\Omega)^3$, the operators curl and div are defined, respectively, by

$$\begin{aligned} \langle \text{curl } u, \varphi \rangle &= \int_{\Omega} u(x) \cdot \text{curl } \varphi(x) \, dx \\ &= \int_{\Omega} \left[u_1(x) \left(\frac{\partial \varphi_3}{\partial x_2} - \frac{\partial \varphi_2}{\partial x_3} \right) + u_2(x) \left(\frac{\partial \varphi_1}{\partial x_3} - \frac{\partial \varphi_3}{\partial x_1} \right) + u_3(x) \left(\frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2} \right) \right] dx, \end{aligned}$$

for every $\varphi \in \mathcal{D}(\Omega)^3$, and

$$\begin{aligned} \langle \text{div } u, \phi \rangle &= - \int_{\Omega} u(x) \cdot \nabla \phi(x) \, dx \\ &= - \int_{\Omega} \left(u_1(x) \frac{\partial \phi}{\partial x_1}(x) + u_2(x) \frac{\partial \phi}{\partial x_2}(x) + u_3(x) \frac{\partial \phi}{\partial x_3}(x) \right) dx, \end{aligned}$$

for every $\phi \in \mathcal{D}(\Omega)$. If Ω has a Lipschitz continuous boundary $\partial\Omega$, and n is the outward normal vector to $\partial\Omega$, then the Green formula

$$\int_{\Omega} \text{curl } u(x) \cdot v(x) \, dx = \int_{\partial\Omega} u \cdot (v \times n) \, dS + \int_{\Omega} u(x) \cdot \text{curl } v(x) \, dx \quad (12)$$

holds for every u and v such that $u, \text{curl } u \in L^{p_h(\cdot)}(\Omega)^3$ and $v, \text{curl } v \in L^{p'_h(\cdot)}(\Omega)^3$.

Another class of function spaces we are interested in is the class of intermediate spaces. Given two Lebesgue spaces $L^p(\Omega)^3$ and $L^q(\Omega)^3$, we define the algebraic sum of both as

$$L^p(\Omega)^3 + L^q(\Omega)^3 = \{w = v_1 + v_2 : v_1 \in L^p(\Omega)^3, v_2 \in L^q(\Omega)^3\},$$

which is itself a Banach space under the norm

$$\|w\|_{L^p(\Omega)^3 + L^q(\Omega)^3} = \inf_{w=v_1+v_2} \{ \|v_1\|_{L^p(\Omega)^3} + \|v_2\|_{L^q(\Omega)^3} \}.$$

See [31]. The space $L_{hom}(\Omega)^3$ of all measurable functions such that the Γ -limit functional (11) is finite satisfies the inclusions

$$L^p(\Omega)^3 \cap L^q(\Omega)^3 \subset L_{hom}(\Omega)^3 \subset L^p(\Omega)^3 + L^q(\Omega)^3.$$

4. Young measures associated with divergence-free fields

In this section, we start by recalling the definition of Young measure associated with a bounded sequence of functions in a Lebesgue space since it is the main tool used in this work to study the Γ -convergence of variable exponent power law functionals. Then, we comment some properties of a special type of Young measure: the one associated with sequences of divergence-free functions. For more details see the monographs [33–35] and the articles [36,37].

Let $\{v_h\}$ be a bounded sequence in $L^p(\Omega)^d$, with $p > 1$ and $\Omega \subset \mathbb{R}^n$ a bounded domain. According to the Fundamental Theorem of Young measures, there exist a subsequence (not relabeled) $\{v_h\}$ and a family of probability measures $\{\nu_x\}_{x \in \Omega}$ on $\mathbb{R}^d$ such that, for any Carathéodory function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ bounded from below, it holds

$$\lim_{h \rightarrow \infty} \int_{\Omega} f(x, v_h(x)) \, dx = \int_{\Omega} \int_{\mathbb{R}^d} f(x, \rho) \, dv_x(\rho) \, dx$$

if and only if the sequence $\{f(\cdot, v_h(\cdot))\}$ is equi-integrable, see [38]. This family of probability measures $\{v_x\}_{x \in \Omega}$ is called the Young measure associated with the sequence of functions $\{v_h\}$, and its main feature is the characterization of the weak limit of the sequence $\{f(\cdot, v_h(\cdot))\}$ in $L^1(\Omega)$. Namely, for any bounded sequence $\{v_h\}$ in $L^p(\Omega)^d$ and any continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $|f(\rho)| \leq \alpha(1 + |\rho|^m)$, with $0 < m < p$, there exists a family of probability measures $\{v_x\}_{x \in \Omega}$ so that the sequence of compositions $\{f(v_h(\cdot))\}$ converges weakly to the function $\langle v_x, f \rangle = \int_{\mathbb{R}^d} f(\rho) \, dv_x(\rho)$ in $L^{p/m}(\Omega)$. The barycenter (i.e., the first moment) of a Young measure $\{v_x\}_{x \in \Omega}$ is the map $\langle v, id \rangle$ defined by $\langle v_x, id \rangle = \int_{\mathbb{R}^d} \rho \, dv_x(\rho)$ for a.e. $x \in \Omega$. The following proposition gives a lower bound for the lower limit in terms of the Young measure associated with the sequence $\{v_h\}$ (see [34, Theorem 6.11]).

Proposition 4.1: *Let $\{v_x\}_{x \in \Omega}$ be the Young measure associated with the sequence of functions $v_h : \Omega \rightarrow \mathbb{R}^d$. Then, it holds*

$$\liminf_{h \rightarrow \infty} \int_D f(x, v_h(x)) \, dx \geq \int_D \int_{\mathbb{R}^d} f(x, \rho) \, dv_x(\rho) \, dx,$$

for every Carathéodory function $f : D \times \mathbb{R}^d \rightarrow \mathbb{R}$ bounded from below, and every measurable subset $D \subset \Omega$.

Lemma 4.2 (Riemann–Lebesgue): *Let f be a D -periodic function in $L^\infty(D)$. Set $f_h(x) = f(hx)$ for a.e. $x \in \mathbb{R}^n$. Then, the sequence $\{f_h\}$ converges weakly*, as h goes to $+\infty$, to $\frac{1}{|D|} \int_D f(y) \, dy$ in $L^\infty(\mathbb{R}^n)$.*

An interesting type of Young measure is the homogeneous Young measure concentrated in two points, which comes from the convex combination of two Dirac measures, e.g.

$$v = \alpha \delta_{\Lambda_1} + (1 - \alpha) \delta_{\Lambda_2}, \quad (13)$$

for some constant $\alpha \in (0, 1)$, and $\Lambda_1, \Lambda_2 \in \mathbb{R}^d$, see [33, 34]. If we consider the sequence of functions $v_h : \Omega \rightarrow \mathbb{R}^d$ defined by

$$v_h(x) = \Lambda_1 \chi_{\Omega_\alpha}(hx) + \Lambda_2 (1 - \chi_{\Omega_\alpha}(hx)), \quad (14)$$

where Ω_α is a subset of Ω such that $|\Omega_\alpha| = \alpha|\Omega|$, and χ_{Ω_α} is the characteristic function of the set Ω_α , then we realize that its associated Young measure is v . Indeed, it follows from the Riemann–Lebesgue Lemma that the sequence of characteristic functions $\chi_{\Omega_\alpha}(h \cdot)$ converges weakly*, as h goes to $+\infty$, to α in $L^\infty(\Omega)$ (see [3, Theorem 2.6]).

We are interested in homogeneous Young measures like v in (13) which are associated with sequences of divergence-free functions. So, let us take into account the following lemma (see [34, Lemma 10.4]).

Lemma 4.3: *Let χ_D be the characteristic function of a Lipschitz subdomain $D \subset \Omega \subset \mathbb{R}^d$. Let $\{v_h\}$ be a sequence of functions such that it converges weakly* to v in $L^\infty(\Omega)^d$, and the sequence $\{\operatorname{div} v_h\}$ converges strongly to $\operatorname{div} v$ in $H^{-1}(D)$. Then, the sequence $\{\operatorname{div} \chi_D v_h\}$ converges strongly to $\operatorname{div} \chi_D v$ in $H^{-1}(\Omega)$. In particular, the sequence $\{\operatorname{div} \chi_D v_h\}$ converges strongly to 0 in $H^{-1}(\Omega)$ if $v \cdot e = 0$ on $\partial D \cap \Omega$, where e is the unit normal to D .*

Notice that, if $\Omega \subset \mathbb{R}^d$, the sequence $\{v_h\}$ in (14) is such that $\operatorname{div} v_h = \operatorname{div} ((\Lambda_1 - \Lambda_2) \chi_{\Omega_\alpha})$ in Ω . Thus, if we assume that $(\Lambda_1 - \Lambda_2) \cdot e = 0$ on $\partial \Omega_\alpha \cap \Omega$, where e is the unit normal to Ω_α , we conclude that $\operatorname{div} v_h = 0$ in Ω .

General Young measures associated with sequences of divergence-free functions may be characterized as follows (see [37, Theorem 4.1]).

Theorem 4.4: *Let $1 \leq p < \infty$, and $\{v_x\}_{x \in \Omega}$ be a weakly measurable family of probability measures on $\mathbb{R}^d$. There exists a p -equi-integrable sequence $\{v_h\}$ in $L^p(\Omega)^d$, such that $\operatorname{div} v_h = 0$ in Ω , associated with the Young measure $\{v_x\}_{x \in \Omega}$ if and only if*

- (1) *there exists $v \in L^p(\Omega)^d$ such that $\operatorname{div} v = 0$ in Ω , and $v(x) = \int_{\mathbb{R}^d} \rho \, dv_x(\rho)$ for a.e. $x \in \Omega$;*
- (2) $\int_{\Omega} \int_{\mathbb{R}^d} |\rho|^p \, dv_x(\rho) \, dx < +\infty$.

The proposition below gives a useful result on the approximation of sequences of divergence-free functions (see [37, Lemma 2.17] and [36, Proposition 2.3]).

Proposition 4.5: *Let $\{v_h\}$ be a sequence that converges weakly* to v in $L^\infty(\Omega)^d$, so that the sequence $\{\operatorname{div} v_h\}$ converges weakly to 0 in $L^p(\Omega)$ for some $p > d$. Let $v = \{v_x\}_{x \in \Omega}$ be the Young measure associated with $\{v_h\}$. Then, there exists a sequence $\{u_h\} \subset L^\infty(\Omega)^d$ such that $\operatorname{div} u_h = 0$ in Ω , $\int_{\Omega} u_h(x) \, dx = \int_{\Omega} v(x) \, dx$, and the sequence $\{u_h - v_h\}$ converges strongly to 0 in $L^\infty(\Omega)^d$. In particular, the sequence $\{u_h\}$ generates the Young measure v .*

5. Γ -convergence of variable exponent power law functionals

The notion of Γ -convergence of functionals defined in variable exponent Lebesgue spaces we are interested in reads as follows.

Definition 1: We say that the sequence of functionals $F_h : L^{p_h(\cdot)}(\Omega)^3 \rightarrow \mathbb{R} \cup \{\infty\}$ is Γ -convergent, with respect to the strong topology in $L^p(\Omega)^3$, to the functional $F : L_{hom}(\Omega)^3 \rightarrow \mathbb{R} \cup \{\infty\}$ if, for any $u \in L_{hom}(\Omega)^3$,

- (1) for every sequence $\{u_h\} \subset L^{p_h(\cdot)}(\Omega)^3$ such that $u_h \rightarrow u$ strongly in $L^p(\Omega)^3$, then

$$F(u) \leq \liminf_{h \rightarrow \infty} F_h(u_h);$$

- (2) there exists a sequence $\{u_h\} \subset L^{p_h(\cdot)}(\Omega)^3$ such that $u_h \rightarrow u$ strongly in $L^p(\Omega)^3$, and

$$F(u) = \lim_{h \rightarrow \infty} F_h(u_h).$$

The choice of the topology associated with the Banach space plays an important role in proving the Γ -convergence of sequences of functionals. Given the sequence of functionals F_h in (9) and the sequence of powers p_h in (5), if we consider a sequence $\{u_h\} \subset X^{p_h(\cdot)}(\Omega)^3$ such that the sequence $\{F_h(u_h)\}$ is bounded, then there exists a weak convergent subsequence of $\{u_h\}$ in $X^p(\Omega)^3$.

Now, we will present the proof of the main theorem.

Proof of Theorem 2.1: The proof is divided into two parts. Firstly, we prove that the functional F in (11) is a lower bound for the sequence of functionals F_h in (9); and finally we prove there exists a recovering sequence for which the lower bound is attained.

1st Part. Let $u \in L_{hom}(\Omega)^3$ be such that $F(u) < +\infty$; otherwise, there is nothing to prove. Take a sequence of functions $u_h \in X^{p_h(\cdot)}(\Omega)^3$ such that it converges strongly to u in $L^p(\Omega)^3$, and consider the sequence of powers $\{p_h\}$ defined in (5). Let $\{v_x\}_{x \in \Omega}$ be the Young measure associated with the sequence of pairs $\{(p_h, \operatorname{curl} u_h)\}$. It follows from Proposition 4.1 that

$$\liminf_{h \rightarrow \infty} \int_{\Omega} f(p_h(x), \operatorname{curl} u_h(x)) \, dx \geq \int_{\Omega} \int_{\mathbb{R}^4} f(\lambda, \rho) \, dv_x(\lambda, \rho) \, dx.$$

Notice that the sequence $\{p_h\}$ generates the homogeneous Young measure $\sigma = \alpha\delta_p + (1 - \alpha)\delta_q$ supported on $\{p, q\}$. In this way, for a.e. $x \in \Omega$, there exist probability measures $\mu_{p,x}$ and $\mu_{q,x}$ supported on $\mathbb{R}^3$ such that v_x may be decomposed as $v_x(\lambda, \rho) = \alpha\mu_{p,x}(\rho) \otimes \delta_p(\lambda) + (1 - \alpha)\mu_{q,x}(\rho) \otimes \delta_q(\lambda)$. This implies

$$\begin{aligned} & \int_{\Omega} \int_{\mathbb{R}^4} f(\lambda, \rho) \, dv_x(\lambda, \rho) \, dx \\ &= \int_{\Omega} \left(\alpha \int_{\mathbb{R}^3} f(p, \rho) \, d\mu_{p,x}(\rho) + (1 - \alpha) \int_{\mathbb{R}^3} f(q, \rho) \, d\mu_{q,x}(\rho) \right) dx. \end{aligned}$$

We may define the function $\varphi : \{p, q\} \times \Omega \rightarrow \mathbb{R}^3$ by putting

$$\varphi(\lambda, x) = \int_{\mathbb{R}^3} \rho \, d\mu_{\lambda,x}(\rho),$$

so that

$$\begin{aligned} & \int_{\Omega} \left(\alpha \int_{\mathbb{R}^3} f(p, \rho) \, d\mu_{p,x}(\rho) + (1 - \alpha) \int_{\mathbb{R}^3} f(q, \rho) \, d\mu_{q,x}(\rho) \right) dx \\ & \geq \int_{\Omega} (\alpha f(p, \varphi(p, x)) + (1 - \alpha) f(q, \varphi(q, x))) \, dx \end{aligned}$$

thanks to the Jensen inequality and the convexity of $f(\lambda, \cdot)$ in $\mathbb{R}^3$, for any $\lambda \in \{p, q\}$. Thus,

$$\liminf_{h \rightarrow \infty} \int_{\Omega} f(p_h(x), \operatorname{curl} u_h(x)) \, dx \geq \int_{\Omega} (\alpha f(p, \varphi(p, x)) + (1 - \alpha) f(q, \varphi(q, x))) \, dx.$$

Let $\{\theta_x\}_{x \in \Omega}$ be the Young measure associated with the sequence $\{\operatorname{curl} u_h\}$ supported on $\mathbb{R}^3$. Then, for a.e. $x \in \Omega$, we may characterize θ_x as $\theta_x(\rho) = \alpha\mu_{p,x}(\rho) + (1 - \alpha)\mu_{q,x}(\rho)$, whose barycenter is

$$\operatorname{curl} u(x) = \alpha\varphi(p, x) + (1 - \alpha)\varphi(q, x).$$

Recall that the sequence $\{p_h\}$ is a two-phase sequence, i.e. it takes two different values, p and q , depending on the phase. We have assumed that each phase is normal to the unit vector e . The probability measure θ_x is a homogeneous Young measure associated with a sequence of divergence-free piecewise-constant functions whenever it holds $(\varphi(p, x) - \varphi(q, x)) \cdot e = 0$.

Now, from the arbitrariness of φ , it follows

$$\int_{\Omega} (\alpha f(p, \varphi(p, x)) + (1 - \alpha) f(q, \varphi(q, x))) \, dx \geq \int_{\Omega} f_{hom}(\operatorname{curl} u(x)) \, dx,$$

where we have defined the function $f_{hom} : \mathbb{R}^3 \rightarrow \mathbb{R}$ by putting

$$f_{hom}(\rho) = \min_{\Lambda_1, \Lambda_2 \in \mathbb{R}^3} \left\{ \alpha f(p, \Lambda_1) + (1 - \alpha) f(q, \Lambda_2) : \begin{array}{l} \rho = \alpha \Lambda_1 + (1 - \alpha) \Lambda_2 \\ (\Lambda_1 - \Lambda_2) \cdot e = 0 \end{array} \right\}.$$

Therefore, we conclude

$$\liminf_{h \rightarrow \infty} \int_{\Omega} f(p_h(x), \operatorname{curl} u_h(x)) \, dx \geq \int_{\Omega} f_{hom}(\operatorname{curl} u(x)) \, dx.$$

2nd Step: Since $\mathcal{D}(\Omega)^3$ is dense in $L^{p_h(\cdot)}(\Omega)^3$, let us consider $u \in \mathcal{D}(\Omega)^3$ such that $F(u) < +\infty$. Provided Ω is an open bounded set, we know that, for each $i \in \mathbb{N}$, there exists a family of points

$\{x_i^k\} \subset \Omega \setminus N_i$, with $|N_i| = 0$, and positive numbers $\{r_i^k\}$, $r_i^k \leq 1/i$, such that

$$\Omega = \bigcup_k (x_i^k + r_i^k \Omega) \bigcup N_i$$

and

$$\int_{\Omega} f_{hom}(\operatorname{curl} u(x)) \, dx = \lim_{i \rightarrow \infty} \sum_k f_{hom}(\operatorname{curl} u(x_i^k)) \left| x_i^k + r_i^k \Omega \right|,$$

see [34, Lemma 7.9]. It follows from the definition of f_{hom} that, for each x_i^k , there exists an optimal pair $(\Lambda_1(x_i^k), \Lambda_2(x_i^k)) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$f_{hom}(\operatorname{curl} u(x_i^k)) = \alpha f(p, \Lambda_1(x_i^k)) + (1 - \alpha) f(q, \Lambda_2(x_i^k)),$$

with $\operatorname{curl} u(x_i^k) = \alpha \Lambda_1(x_i^k) + (1 - \alpha) \Lambda_2(x_i^k)$ and $(\Lambda_1(x_i^k) - \Lambda_2(x_i^k)) \cdot e = 0$. Thus, we have

$$\int_{\Omega} f_{hom}(\operatorname{curl} u(x)) \, dx = \lim_{i \rightarrow \infty} \sum_k \left(\alpha f(p, \Lambda_1(x_i^k)) + (1 - \alpha) f(q, \Lambda_2(x_i^k)) \right) \left| x_i^k + r_i^k \Omega \right|.$$

For each $i, k \in \mathbb{N}$, if we define the sequence of pairs $\{(p_h, V_h^{i,k})\}$ in Ω by putting

$$(p_h(y), V_h^{i,k}(y)) = (p, \Lambda_1(x_i^k)) \chi_{(0,\alpha)}(hy \cdot e) + (q, \Lambda_2(x_i^k)) (1 - \chi_{(0,\alpha)}(hy \cdot e)),$$

then it follows from the Riemann–Lebesgue Lemma that $\{(p_h, V_h^{i,k})\}$ converges weakly*, as h goes to $+\infty$, to $(\alpha p + (1 - \alpha)q, \alpha \Lambda_1(x_i^k) + (1 - \alpha) \Lambda_2(x_i^k))$ in $L^\infty(\Omega)^4$. Notice that for each $h, i, k \in \mathbb{N}$, $\operatorname{div} V_h^{i,k} = \operatorname{div} (\Lambda_1(x_i^k) - \Lambda_2(x_i^k)) \chi_{(0,\alpha)} = 0$ in Ω . Moreover,

$$\lim_{h \rightarrow \infty} \frac{1}{|\Omega|} \int_{\Omega} f(p_h(y), V_h^{i,k}(y)) \, dy = \alpha f(p, \Lambda_1(x_i^k)) + (1 - \alpha) f(q, \Lambda_2(x_i^k)).$$

In this way,

$$\int_{\Omega} f_{hom}(\operatorname{curl} u(x)) \, dx = \lim_{i \rightarrow \infty} \lim_{h \rightarrow \infty} \sum_k \int_{x_i^k + r_i^k \Omega} f \left(p_h \left(\frac{x - x_i^k}{r_i^k} \right), V_h^{i,k} \left(\frac{x - x_i^k}{r_i^k} \right) \right) \, dx.$$

Let us consider a sequence of smooth cut-off functions $g_m \in C_0^\infty(\Omega)$ such that $g_m(y) = 1$ if $y \in \Omega_m^1 = \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) \geq 1/m\}$, $g_m(y) = 0$ if $y \in \Omega_m^0 = \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) \leq 1/2m\}$, and there exists a constant $c > 0$ for which $\|\nabla g_m\|_{L^\infty(\Omega)^3} \leq c m$. Now, for each $h, i, m \in \mathbb{N}$, we may define the sequence of functions $Z_{h,i,m}$ in the domain Ω by putting

$$Z_{h,i,m}(x) = V_h^{i,k} \left(\frac{x - x_i^k}{r_i^k} \right) g_m \left(\frac{x - x_i^k}{r_i^k} \right) + \operatorname{curl} u(x) \left(1 - g_m \left(\frac{x - x_i^k}{r_i^k} \right) \right)$$

if $x \in x_i^k + r_i^k \Omega$. Since $f(\lambda, \cdot)$ is convex in $\mathbb{R}^3$, for every $\lambda \in \mathbb{R}$, it follows

$$\begin{aligned} & \int_{\Omega} f(p_h(x), Z_{h,i,m}(x)) \, dx \\ & \leq \sum_k \int_{x_i^k + r_i^k \Omega} g_m \left(\frac{x - x_i^k}{r_i^k} \right) f \left(p_h(x), V_h^{i,k} \left(\frac{x - x_i^k}{r_i^k} \right) \right) \, dx \\ & \quad + \sum_k \int_{x_i^k + r_i^k \Omega} \left(1 - g_m \left(\frac{x - x_i^k}{r_i^k} \right) \right) f(p_h(x), \operatorname{curl} u(x)) \, dx. \end{aligned}$$

We have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \lim_{i \rightarrow \infty} \lim_{h \rightarrow \infty} \sum_k \int_{x_i^k + r_i^k \Omega} g_m \left(\frac{x - x_i^k}{r_i^k} \right) f \left(p_h(x), V_h^{i,k} \left(\frac{x - x_i^k}{r_i^k} \right) \right) \, dx \\ & \leq \lim_{i \rightarrow \infty} \lim_{h \rightarrow \infty} \sum_k \int_{x_i^k + r_i^k \Omega} f \left(p_h(x), V_h^{i,k} \left(\frac{x - x_i^k}{r_i^k} \right) \right) \, dx = \int_{\Omega} f_{hom}(\operatorname{curl} u(x)) \, dx \end{aligned}$$

and

$$\begin{aligned} & \lim_{m \rightarrow \infty} \lim_{i \rightarrow \infty} \lim_{h \rightarrow \infty} \sum_k \int_{x_i^k + r_i^k \Omega} \left(1 - g_m \left(\frac{x - x_i^k}{r_i^k} \right) \right) f(p_h(x), \operatorname{curl} u(x)) \, dx \\ & = \lim_{m \rightarrow \infty} \lim_{i \rightarrow \infty} \lim_{h \rightarrow \infty} \sum_k \int_{x_i^k + r_i^k \Omega \setminus \Omega_m^1} f(p_h(x), \operatorname{curl} u(x)) \, dx \\ & = \lim_{m \rightarrow \infty} \lim_{i \rightarrow \infty} \sum_k \int_{x_i^k + r_i^k \Omega \setminus \Omega_m^1} (\alpha f(p, \operatorname{curl} u(x)) + (1 - \alpha) f(q, \operatorname{curl} u(x))) \, dx \\ & = \lim_{m \rightarrow \infty} \int_{\Omega \setminus \Omega_m^1} (\alpha f(p, \operatorname{curl} u(x)) + (1 - \alpha) f(q, \operatorname{curl} u(x))) \, dx = 0, \end{aligned}$$

cause the sequence $\{p_h\}$ converges weakly* to $\alpha p + (1 - \alpha)q$ in $L^\infty(\Omega)$, and the measure $|\Omega \setminus \Omega_m^1|$ vanishes as m goes to $+\infty$. Thus,

$$\lim_{m \rightarrow \infty} \lim_{i \rightarrow \infty} \lim_{h \rightarrow \infty} \int_{\Omega} f(p_h(x), Z_{h,i,m}(x)) \, dx \leq \int_{\Omega} f_{hom}(\operatorname{curl} u(x)) \, dx.$$

Now, by a diagonalization process, let us consider a subsequence $\{Z_h\} = \{Z_{h,i(h),m(h)}\}$ such that

$$\begin{aligned} \limsup_{h \rightarrow \infty} \int_{\Omega} f(p_h(x), Z_h(x)) \, dx & \leq \lim_{m \rightarrow \infty} \lim_{i \rightarrow \infty} \lim_{h \rightarrow \infty} \int_{\Omega} f(p_h(x), Z_{h,i,m}(x)) \, dx \\ & \leq \int_{\Omega} f_{hom}(\operatorname{curl} u(x)) \, dx. \end{aligned}$$

Since the sequence $\{Z_{h,i,m}\}$ converges weakly* to $\operatorname{curl} u$ in $L^\infty(\Omega)^3$, and the sequence $\{\operatorname{div} Z_{h,i,m}\}$ converges weakly to 0 in $L^r(\Omega)^3$ with $r > 3$, it follows from Proposition 4.5 that there exists a sequence $\{U_h\}$ in $L^\infty(\Omega)^3$ such that $\operatorname{div} U_h = 0$ in Ω , for every $h \in \mathbb{N}$, and the sequences $\{U_h\}$ and $\{Z_h\}$ generate the same Young measure. Besides, there exists a sequence of functions u_h in $L^\infty(\Omega)^3$ such that $\operatorname{curl} u_h = U_h$ in Ω , $\operatorname{div} u_h = 0$ in Ω , and $u_h \cdot n = 0$ on $\partial\Omega$, where n stands for the unit

outward normal vector to $\partial\Omega$, provided $\operatorname{div} U_h = 0$ in Ω (see [23, Theorem 3.5]). In this way, it holds

$$\begin{aligned} \limsup_{h \rightarrow \infty} \int_{\Omega} f(p_h(x), \operatorname{curl} u_h(x)) \, dx &= \limsup_{h \rightarrow \infty} \int_{\Omega} f(p_h(x), Z_h(x)) \, dx \\ &\leq \int_{\Omega} f_{\text{hom}}(\operatorname{curl} u(x)) \, dx. \end{aligned}$$

□

6. Homogenization of $p_h(x)$ -curl systems with examples

The notion of Γ -convergence is intimately related to the convergence of families of minimizing problems. Namely, if a sequence of functionals Γ -converges to a limit functional, and these functionals have minimizers, then there exists a subsequence of minimizers which converges, in the appropriated topology, to a minimizer of the Γ -limit functional (see [15, Theorem 1.21]).

Let us consider the boundary value problem

$$\begin{cases} \operatorname{curl} \left(\mu_h^{-1}(x) |\operatorname{curl} u(x)|^{p_h(x)-2} \operatorname{curl} u(x) \right) = \operatorname{curl} j(x) & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega \\ \mu_h^{-1} |\operatorname{curl} u|^{p_h-2} \operatorname{curl} u \times n = j \times n & \text{on } \partial\Omega, \end{cases} \quad (15)$$

where the sequences $\{\mu_h\}$ and $\{p_h\}$ are defined in (1) and (5), respectively, which turn out to be the first-order optimality condition associated with the power law functional E_h in (16).

Proposition 6.1: *If $u_h \in X^{p_h(\cdot)}(\Omega)^3$ is a minimizer of the energy*

$$E_h(u) = \int_{\Omega} \left(\frac{\mu_h^{-1}(x)}{p_h(x)} |\operatorname{curl} u(x)|^{p_h(x)} - j(x) \cdot \operatorname{curl} u(x) \right) dx, \quad (16)$$

then u_h is a solution of the boundary value problem (15).

Proof: If the function $u_h \in X^{p_h(\cdot)}(\Omega)^3$ is a minimizer of the energy E_h in (16), then

$$\left. \frac{d}{ds} E_h(u_h + s\varphi) \right|_{s=0} = 0$$

for every $\varphi \in X^{p_h(\cdot)}(\Omega)^3$. This is equivalent to have

$$\int_{\Omega} \left(\mu_h^{-1}(x) |\operatorname{curl} u(x)|^{p_h(x)-2} \operatorname{curl} u(x) \cdot \operatorname{curl} \varphi(x) - j(x) \cdot \operatorname{curl} \varphi(x) \right) dx = 0.$$

Now, we may apply the Green formula (12) so that the previous equation reduces to

$$\begin{aligned} &\int_{\Omega} \left(\operatorname{curl} \left(\mu_h^{-1}(x) |\operatorname{curl} u(x)|^{p_h(x)-2} \operatorname{curl} u(x) \right) - \operatorname{curl} j(x) \right) \cdot \varphi(x) \, dx \\ &+ \int_{\partial\Omega} \left(\mu_h^{-1}(x) |\operatorname{curl} u(x)|^{p_h(x)-2} \operatorname{curl} u(x) \times n - j(x) \times n \right) \cdot \varphi(x) \, dS = 0. \end{aligned}$$

From the arbitrariness of φ , we conclude that u_h is solution of the system (15). □

Therefore, we may apply Theorem 2.1 to the sequence of power law functionals E_h in (16), and conclude that it Γ -converges, with respect to the strong convergence in $L^p(\Omega)^3$, to the energy E

defined by

$$E(u) = \int_{\Omega} (\psi(\operatorname{curl} u(x)) - j(x) \cdot \operatorname{curl} u(x)) \, dx$$

for any $u \in X^p(\Omega)^3$ such that $\operatorname{curl} u \in L_{hom}(\Omega)^3$. The density $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ is given by

$$\psi(\rho) = \min_{\Lambda_1, \Lambda_2 \in \mathbb{R}^3} \left\{ \alpha \frac{\mu_I^{-1}}{p} |\Lambda_1|^p + (1 - \alpha) \frac{\mu_{II}^{-1}}{q} |\Lambda_2|^q : \begin{array}{l} \rho = \alpha \Lambda_1 + (1 - \alpha) \Lambda_2 \\ (\Lambda_1 - \Lambda_2) \cdot e = 0 \end{array} \right\}.$$

Thus, the first-order optimality condition associated with the Γ -limit functional E is the following boundary value problem

$$\left\{ \begin{array}{ll} \operatorname{curl} (\nabla \psi (\operatorname{curl} u(x))) = \operatorname{curl} j(x) & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega \\ \nabla \psi (\operatorname{curl} u) \times n = j \times n & \text{on } \partial\Omega, \end{array} \right.$$

which turns out to be the homogenized problem associated with the family of boundary value problems in (15).

Example 1: In the simplest case, when $p = 2 = q$, $\alpha = \frac{1}{2}$, $\mu_I = 1$, $\mu_{II} = \frac{1}{2}$ and $e = (1, 0, 0)$, the boundary value problem in (15) reads as

$$\left\{ \begin{array}{ll} \operatorname{curl} (\mu_h^{-1}(x) \operatorname{curl} u(x)) = \operatorname{curl} j(x) & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega \\ \mu_h^{-1} \operatorname{curl} u \times n = j \times n & \text{on } \partial\Omega, \end{array} \right.$$

and its associated energy

$$E_h(u) = \int_{\Omega} \left(\frac{\mu_h^{-1}(x)}{2} |\operatorname{curl} u(x)|^2 - j(x) \cdot \operatorname{curl} u(x) \right) dx$$

if $u \in X^2(\Omega)^3$. Therefore, the sequence $\{E_h\}$ is Γ -convergent to the functional E whose density $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined by $\psi(\rho) = \frac{1}{2} M \rho \cdot \rho$, where the effective matrix M is given by

$$M = \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix},$$

and the homogenized problem is

$$\left\{ \begin{array}{ll} \operatorname{curl} (M \operatorname{curl} u(x)) = \operatorname{curl} j(x) & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega \\ M \operatorname{curl} u \times n = j \times n & \text{on } \partial\Omega. \end{array} \right.$$

Example 2: In the case that $p = 2, q = \frac{5}{2}, \alpha = \frac{1}{2}, \mu_I = \mu_{II} = 1$ and $e = (1, 0, 0)$, we are considering the boundary value problem

$$\begin{cases} \operatorname{curl}(|\operatorname{curl} u(x)|^{p_h(x)-2} \operatorname{curl} u(x)) = \operatorname{curl} j(x) & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega \\ |\operatorname{curl} u|^{p_h-2} \operatorname{curl} u \times n = j \times n & \text{on } \partial\Omega, \end{cases}$$

and the associated energy

$$E_h(u) = \int_{\Omega} \left(\frac{1}{p_h(x)} |\operatorname{curl} u(x)|^{p_h(x)} - j(x) \cdot \operatorname{curl} u(x) \right) dx$$

if $u \in X^{p_h(\cdot)}(\Omega)^3$. The sequence $\{E_h\}$ is Γ -convergent to the functional E whose density $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined through a two-dimensional minimization problem with three parameters as follows

$$\psi(\rho_1, \rho_2, \rho_3) = \min_{x, y \in \mathbb{R}} \left\{ \frac{1}{5} (\rho_1^2 + (2\rho_2 - x)^2 + (2\rho_3 - y)^2)^{\frac{5}{4}} + \frac{1}{4} (\rho_1^2 + x^2 + y^2) \right\},$$

which is equivalent to have

$$\psi(\rho_1, \rho_2, \rho_3) = \frac{1}{4} (\rho_1^2 + 4\rho_2^2 + 4\rho_3^2) + \frac{1}{5} (\rho_1^2 + r^2)^{\frac{5}{4}} - r\sqrt{\rho_2^2 + \rho_3^2} + \frac{1}{4} r^2,$$

where r satisfies the equation $r(\rho_1^2 + r^2)^{\frac{1}{4}} + r - \sqrt{\rho_2^2 + \rho_3^2} = 0$. Thus, the Γ -limit E is defined in the intermediate space between $L^2(\Omega)^3$ and $L^{\frac{5}{2}}(\Omega)^3$. Precisely, we have that $\psi(\rho_1, 0, 0) = \frac{1}{5}\rho_1^{\frac{5}{2}} + \frac{1}{4}\rho_1^2$, for every $\rho_1 \in \mathbb{R}$, while $\psi(0, \rho_2, 0)$ and $\psi(0, 0, \rho_3)$ are quadratic polynomials, for every $\rho_2, \rho_3 \in \mathbb{R}$, since

$$\psi(0, \rho_2, 0) = \rho_2^2 - |\rho_2|r + \frac{1}{4}r^2 + \frac{1}{5}r^{\frac{5}{2}},$$

where r is the real solution of $r^{\frac{3}{2}} + r - 2|\rho_2| = 0$, i.e., $r = \left(\frac{1}{9z} + z - \frac{1}{3}\right)^2$ with

$$z = \left(\rho_2 + \sqrt{\left(|\rho_2| - \frac{1}{27}\right)^2 - \frac{1}{729} - \frac{1}{27}} \right)^{\frac{1}{3}},$$

as well as

$$\psi(0, 0, \rho_3) = \rho_3^2 - |\rho_3|r + \frac{1}{4}r^2 + \frac{1}{5}r^{\frac{5}{2}},$$

where r is the real solution of $r^{\frac{3}{2}} + r - 2|\rho_3| = 0$.

Example 3: In the case that $p = 2, q = 3, \alpha = \frac{1}{3}, \mu_I = \frac{1}{2}, \mu_{II} = \frac{1}{3}, e = (1, 0, 0)$, the sequence of energies $\{E_h\}$ is Γ -convergent to the functional E whose density $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined by

$$\psi(\rho_1, \rho_2, \rho_3) = \min_{x, y \in \mathbb{R}} \left\{ \frac{1}{3} (\rho_1^2 + x^2 + y^2) + \frac{2}{3} \left(\rho_1^2 + \left(\frac{3\rho_2 - x}{2}\right)^2 + \left(\frac{3\rho_3 - y}{2}\right)^2 \right)^{\frac{3}{2}} \right\},$$

which is equivalent to

$$\psi(\rho_1, \rho_2, \rho_3) = \frac{1}{3} (\rho_1^2 + 9\rho_2^2 + 9\rho_3^2) + \frac{1}{3} r^2 + \frac{2}{3} \left(\rho_1^2 + \frac{r^2}{4} \right)^{\frac{3}{2}} - 2r\sqrt{\rho_2^2 + \rho_3^2},$$

where r satisfies the equation $r \left(\rho_1^2 + \frac{r^2}{4} \right)^{\frac{1}{2}} + \frac{4}{3}r - 4\sqrt{\rho_2^2 + \rho_3^2} = 0$. In this way, the Γ -limit E is defined in the intermediate space between $L^2(\Omega)^3$ and $L^3(\Omega)^3$. Namely, we have that $\psi(\rho_1, 0, 0) = \frac{1}{3}\rho_1^3 + \frac{2}{3}\rho_1^2$, for every $\rho_1 \in \mathbb{R}$, while $\psi(0, \rho_2, 0)$ and $\psi(0, 0, \rho_3)$ are quadratic polynomials, for every $\rho_2, \rho_3 \in \mathbb{R}$, provided

$$\psi(0, \rho_2, 0) = 3\rho_2^2 - \frac{8}{9}|\rho_2|\sqrt{4 + 18|\rho_2|} + \frac{8}{3}|\rho_2| - \frac{16}{81}\sqrt{4 + 18|\rho_2|} + \frac{32}{81}$$

and

$$\psi(0, 0, \rho_3) = 3\rho_3^2 - \frac{8}{9}|\rho_3|\sqrt{4 + 18|\rho_3|} + \frac{8}{3}|\rho_3| - \frac{16}{81}\sqrt{4 + 18|\rho_3|} + \frac{32}{81}.$$

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