Optimal experimental design on the loading frequency for a probabilistic fatigue model for plain and fibre-reinforced concrete

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Abstract

The objective is to improve the fatigue characterisation process based on the concept of optimal experimental design. This is carried out through a probabilistic model, previously developed, which takes into account the experimentally observed loading frequency effect on the fatigue life in plain and fibre-reinforced concrete. The Fisher Information Matrix is first obtained for the simplified fatigue model. The optimal design is found to be located at the minimum values allowed for both the maximum stress and stress ratio, whereas the two loading frequencies are the minimum and maximum values in the defined range. Next, the FIM is derived for the extended fatigue model. The previously carried out experimental plan is 65% efficient compared to the optimum. Even though it has been developed for the specific chosen fatigue model, the current procedure can be applied to any other fatigue model to significantly improve the fatigue characterisation process of any material.

Keywords: Fisher Information Matrix (FIM), D-optimality, Weibull cumulative distribution function, loading frequency, fatigue model

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Highlights

- A general procedure is developed for obtaining the optimal design of a fatigue characterisation process.
- The Fisher Information Matrix is derived for both the simplified and complete fatigue models. The efficiency of the previously carried out experimental design is approximately 87% and 65% for the simplified and complete fatigue model, respectively.
- A robustness analysis is carried out for the obtained optimum. The efficiency surpasses 99% when the initial values fluctuated between 50% and 150% of the corresponding reference values.

1. Introduction

Fatigue tests are known to be time consuming, and can sometimes be unachievable if not properly designed. The procedure on how to determine the number of tests needed to characterise certain materials is an open issue. In the current work, we make use of the Fisher Information Matrix (FIM) to derive the optimal location of tests to characterise fatigue performance of concrete-related materials under given loading conditions. In particular, the fatigue model based on an initial distribution developed by Saucedo *et al.* [1] is chosen as an example to carry out the optimal design process. The developed methodology, however, can be applied to any other given fatigue model.

Development of high-performance concrete for structures undergoing dynamic and cyclic loading has led to experiments conducted to study the influence of different fatigue parameters [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. These parameters are related to either the fatigue test conditions, such as the minimum stress, σ_{min} , the maximum stress, σ_{max} , the loading frequency, f , or the material properties; for example the static material strength, σ_c , (compressive or tensile). In the case of concrete, it has been detected that the influence of the stress ratio, R (σ_{min} divided by σ_{max}), the loading frequency, f , and the stress level, σ_{max}/σ_c , is quite relevant [7, 15, 20, 19].

Although the influence of loading frequency has been observed as early as the 1950s [3], the loading time [13] or frequency [14] was not included in the fatigue equation until the 1980s. Moreover, the static strength in concrete exhibits large dispersion and a Weibull distribution has been considered as the best distribution to fit fatigue life in concrete [15, 17]. Recently Saucedo *et al.* [1] proposed a fatigue model to take into account the fact that, on the one hand, the static strength of concrete is a Weibull distribution; on the other hand, such a distribution can be considered as the limit behaviour when the fatigue life approaches one cycle. Meanwhile, influences of the loading frequency and of the stress ratio have also been incorporated. The application range of the proposed fatigue model is below 10 Hz according to experimental tests by Ruiz *et al.* [21, 19]. Depending on the effect of the loading frequency on the hardening exponent of the dynamic material strength is ignored or not, correspondingly the fatigue model is viewed as *simplified* or *complete*. We aim to explore the procedure to calibrate the model parameters based on the concept of optimal experimental design herein.

Optimal experimental designs are especially useful when experimentation is expensive, time consuming or difficult to carry out. A good design will definitely save time, money and provide a better fitting of the model. An experimental design means selecting experimental conditions in such a way that they result in precise estimates or good predictions with a minimum of sample size, which in other words means adequately choosing the levels of the covariates. The optimality of a design is model-oriented in the sense that a design could be quite good for a particular model and not so good for a rival model. In the case of the existence of rival models a discrimination test may be performed to detect the most appropriate one. Optimal designs can be computed for discrimination purposes [31]. In reliability and survival analysis, the Cox proportional hazards model [22] is quite traditional, although proportionality is not always justified. The model is frequently fitted using partial likelihood. Optimal experimental designs in this context were recently computed by López-Fidalgo and Rivas-López [23]. Even though accelerated failure testing models have been increasingly used in recent years, experimental designs have not been studied as much in this field; see Rivas-López *et al.* [24]. In the current work, optimal experimental design is studied for the aforementioned fatigue model for concrete-like materials; in particular, to best estimate the model parameters under given loading conditions.

The rest of this paper is organised as follows. The concept of optimal experimental design is presented next. Applications to the simplified and complete fatigue model are developed in Section 3 and Section 4, respectively. Relevant conclusions are drawn in Section 5.

2. Optimal experimental design

Let x be the vector of covariates, say an experimental condition (loading frequency, maximum stress and stress ratio for a fatigue test), which can be chosen from a compact *design space*, χ ; typically a product of intervals. For a value of x , a response variable time-to-event, t , (equivalently, the fatigue life, N), is observed. This is considered as a random variable from a parametric family of distributions, indexed by β , a vector of parameters. An *exact design* of size n is defined by a collection of experimental conditions $x_1, x_2, \dots, x_n$ in χ , where some of these may be repeated. Thus, a probability measure can be defined with support on the distinct points of the design with weights proportional to the number of repetitions (replicates). This leads to an extension of this definition to any probability measure, ξ , the so called

74 *approximate design* [26], which will be used in this work. An exact design of
75 a particular size, say n , is what one can only put in practice using just n ex-
76 periments. Extending the concept to any probability measure (approximate
77 design) means a kind of abstraction of the real world. This is made because
78 the concept allows a mathematical result, which proves very useful to com-
79 pute optimal designs. As a matter of fact, computing optimal exact designs
80 is rather difficult and frequently convergence of the typical algorithms can
81 not be proved. The main drawback is that once the optimal approximate
82 design is computed, an exact design has to be obtained using some round-
83 ing procedure. For instance, for a discrete measure, ξ , the experimenter has
84 to perform “approximately” $n_i \approx n\xi(x_i)$ experiments at x_i , in such a way
85 $\sum_i n_i = n$. Imhof *et al.* [30] provided some examples to show that if the
86 sample size is large, any rounding procedure leads to a quite efficient exact
87 design. Otherwise, if n is small, then the impact of the rounding may be
88 quite important. Hereafter n will be the total number of experiments to be
89 performed, while k will be the distinct experimental conditions, some of them
90 replicated. An approximate design with k different points in its support will
91 be frequently called a k -point design.

92 The Fisher information is a way of measuring the amount of information
93 that an observable random variable x contains about an unknown parameter
94 upon which the probability model of x depends. The Fisher Information
95 Matrix (FIM) associated with a finite design, $\xi(x)$, is

$$M(\beta; \xi) = \sum_x I(\beta; x)\xi(x), \quad I(\beta; x) = E \left(-\frac{\partial^2 LL(\beta; t, x)}{\partial \beta \partial \beta'} \right), \quad (1)$$

96 where $LL(\beta; t, x)$ is the log-likelihood function of the observation t for a par-
97 ticular experiment x , $I(\beta; x)$ is the corresponding Fisher Information Matrix,
98 which is the expected value of the negative second derivative of $LL(\beta; t, x)$.
99 In particular, it is well known that restricting to approximate designs with
100 finite support still preserves the whole space of information matrices. This is
101 a consequence of the application of Carathéodory’s theorem. Thus, we may
102 limit ourselves to this type of design.

103 Since the covariance matrix (or dispersion matrix) of the estimators is
104 asymptotically proportional to the inverse of the FIM [25], an optimality
105 criterion will be based on a function of (the inverse of) the FIM. The most
106 popular is D-optimality, which maximises the m^{th} -root of the determinant
107 of the FIM, where m is the number of parameters. Therefore, the D-optimal

108 design is the design maximising the determinant of the FIM. The statistical
 109 interpretation of this criterion is that it minimises the volume of the confi-
 110 dence ellipsoid of the parameters of the model. It is easy to check that if the
 111 support of the D-optimal design is the number of parameters of the model,
 112 which is rather frequent, the determinant is the product of the weights of
 113 the design and of a function depending just on the support points. Then the
 114 product of the weights is maximised by equal weights. Thus, the best of this
 115 type of designs may be computed in an easy way. Consequently the so called
 116 General Equivalence Theorem (GET) [26] can be used to check whether this
 117 is actually the D-optimal design or not. The GET for D-optimality states
 118 that a design ξ_D is D-optimal if and only if

$$\psi(\xi_D, \beta; x) = \text{tr} [M^{-1}(\beta; \xi_D) I(\beta; x)] \leq m, x \in \chi. \quad (2)$$

119 Moreover, equality is reached at the support points of the design.

120 If the model is non-linear then the FIM will depend on the parameters.
 121 Using nominal values of the parameters leads to the so called *locally optimal*
 122 *designs*. The way of using the GET to check D-optimality of a particular
 123 design ξ means computing the FIM for this design, $M(\beta; \xi)$, which, with
 124 specific nominal values of the parameters, will be a matrix with numerical
 125 entries. Then $I(\beta; x)$ will be a matrix depending just on x when the nominal
 126 values of the parameters are used. Thus, for a specific design and nomi-
 127 nal values of the parameters, $\psi(\xi, \beta; x)$ is just a function of x . It may be
 128 checked that this function is non-negative on the design space $x \in \chi$, either
 129 analytically or numerically, e.g. plotting it, if χ is one or two-dimensional.

130 A detailed introduction to the theory of optimal experimental design is
 131 provided, e.g., by Pázman [27], among others. Next, we search such designs
 132 for both the simplified and complete fatigue models.

133 **3. Applications to the simplified fatigue model**

134 In this Section, we first summarise the fatigue model developed by Saucedo
 135 *et al.* [1] for completeness. Then the concept of optimal experimental design
 136 is applied for the case when the effect of loading frequency on the hardening
 137 exponent of the dynamic material strength is ignored.

138 On the one hand, the distribution of the static compressive strength, σ_{f_0} ,
 139 for a quasi-brittle material like concrete is shown to be a three-parameter

140 Weibull distribution [1]

$$F(\sigma_{f_0}) = 1 - \exp \left\{ - \left(\frac{\sigma_{f_0} - \sigma_{min_0}}{\lambda} \right)^K \right\}, \quad (3)$$

141 where the parameters σ_{min_0} (location), λ (scale) and K (shape) are properties
 142 of the material and have to be estimated through static compressive tests.
 143 On the other hand, three limit conditions must be fulfilled for a fatigue test,

$$\begin{aligned} \lim_{N \rightarrow \infty} \sigma_f &= \sigma_{min_0}, \\ \lim_{R \rightarrow 1} \sigma_f &= \sigma_{f_0}, \text{ and} \\ \lim_{N \rightarrow 1} \sigma_f &= \sigma_{f_0}. \end{aligned}$$

144 The first limit condition reflects the fact that when the stress level reaches the
 145 fatigue endurance limit, σ_{min_0} , an infinite number of cycles can be resisted
 146 without causing any damage to the material. The second limit condition
 147 states that, if the stress ratio, R , approaches one, the loading condition is
 148 actually static. Consequently, the material fails at its static strength, σ_{f_0} .
 149 In the same way, the third limit condition emphasises that, if the fatigue
 150 life N is equal to one, the employed maximum stress level is necessarily the
 151 material's static strength, σ_{f_0} . If any of these three conditions is not satisfied
 152 for a fatigue model, the implication is that that model will not reflect physical
 153 reality at asymptotic conditions.

154 The following expression was proposed to relate the equivalent static
 155 strength of the fatigue load with the number of cycles resisted, N ,

$$\sigma_{f_0} = \sigma_{min_0} + (\sigma_{max_0} - \sigma_{min_0}) N^{-[b+c \ln(1+f)](1-R)}, \quad (4)$$

156 where the effect of the loading frequency, f , has been taken into considera-
 157 tion.

158 According to the FIB Code [28], the dynamic stress σ_{max} can be related
 159 to its static counterpart by the equation,

$$\sigma_{max_0} = \sigma_{max} \left(\frac{\dot{\sigma}_0}{2f\Delta\sigma} \right)^{0.014} \equiv \sigma_{max} d^{0.014}, \quad (5)$$

where 0.014 is the dynamic hardening coefficient, $\dot{\sigma}_0$ is the stress rate at
 which static characterisation tests were carried out, whereas

$$d = \frac{\dot{\sigma}_0}{2f\Delta\sigma} = \frac{\dot{\sigma}_0}{2f(1-R)\sigma_{max}}.$$

160 Here $\Delta\sigma$ is the stress range, i.e., $\sigma_{max} - \sigma_{min} = (1 - R)\sigma_{max}$.

By defining the non-dimensional maximum stress, S_{mx} , and endurance limit, S_{m0} , as follows

$$S_{mx} = \frac{\sigma_{max}}{\lambda}, \quad S_{m0} = \frac{\sigma_{min0}}{\lambda} \quad (6)$$

161 and defining two intermediate variables A and B as

$$A = S_{mx}d^{0.014} - S_{m0}, \quad (7)$$

$$B = [b + c \ln(1 + f)](1 - R), \quad (8)$$

162 the cumulative distribution function of N can be rewritten as

$$F(N) = 1 - \exp \{ -A^K N^{BK} \}. \quad (9)$$

163 Thus, once the parameters of the Weibull distribution in Eq. (3) are esti-
164 mated, the distribution of N for a given test condition (where σ_{max} , R and
165 f are specified), depends on the two parameters b and c , which are to be
166 estimated.

167 The likelihood and log-likelihood functions for specific values of the pa-
168 rameter set, (f, σ_{max}, R) , and fatigue life, N , are respectively

$$\begin{aligned} L(b, c) &= \frac{\partial F(N)}{\partial N} = A^K B^K N^{BK-1} \exp \{ -A^K N^{BK} \}, \\ LL(b, c) &= K \ln(A) + \ln(B) + \ln(K) + (BK - 1) \ln(N) - A^K N^{BK}. \end{aligned}$$

169 Since

$$\begin{aligned} \frac{\partial B}{\partial b} &= (1 - R), \\ \frac{\partial B}{\partial c} &= (1 - R) \ln(1 + f) = \frac{\partial B}{\partial b} \ln(1 + f), \end{aligned}$$

170 then

$$\begin{aligned} \frac{\partial^2 LL}{\partial b^2} &= -\frac{(1 - R)^2}{B^2} - A^K N^{BK} K^2 (1 - R)^2 [\ln(N)]^2, \\ \frac{\partial^2 LL}{\partial b \partial c} &= \frac{\partial^2 LL}{\partial b^2} \ln(1 + f), \\ \frac{\partial^2 LL}{\partial c^2} &= \frac{\partial^2 LL}{\partial b^2} [\ln(1 + f)]^2, \end{aligned}$$

171 and the first element of the FIM, $I(b, c; f, \sigma_{max}, R)$, is

$$\begin{aligned} I_{11}(b, c; f, \sigma_{max}, R) &= E \left[-\frac{\partial^2 LL}{\partial b^2} \right] = \int_0^\infty \left(-\frac{\partial^2 LL}{\partial b^2} \right) L(N) dN \\ &= \frac{(\pi^2/6) + [1 - C_{EM} - K \ln(A)]^2}{[b + c \ln(1 + f)]^2}, \end{aligned}$$

where E stands for the mathematical expectation according to the probability distribution of the data; that is, the integral of the quantity multiplied by the density function (or the likelihood function); and

$$C_{EM} = \lim_{m \rightarrow \infty} \sum_{i=1}^m \left(\frac{1}{i} - \ln(m) \right) \simeq 0.577216$$

172 represents the Euler-Masheroni constant [29]. For simplicity, we omit the ar-
 173 guments of the ij -th component of the FIM, $I(b, c; f, \sigma_{max}, R)$, by denoting it
 174 as I_{ij} . In addition, by defining $X(f) = \log(1 + f)$, the remaining components
 175 are written as follows

$$I_{12} = I_{21} = I_{11}X(f), \quad I_{22} = I_{11}[X(f)]^2.$$

Therefore the FIM

$$I(b, c; f, \sigma_{max}, R) = \phi \cdot \phi^T,$$

where

$$\phi = I_{11} \begin{pmatrix} 1 \\ X(f) \end{pmatrix},$$

which is the FIM of a linear model with these regressors. If the D-optimal design has two points, then their weights must be equal, as stated in the Introduction. We will consider a 2-point approximate design of this type,

$$\xi = \left\{ \begin{pmatrix} f_1, \sigma_{max1}, R_1 \\ 1/2 \end{pmatrix} \begin{pmatrix} f_2, \sigma_{max2}, R_2 \\ 1/2 \end{pmatrix} \right\}.$$

The FIM is

$$M(b, c; \xi) = \frac{1}{2} \begin{pmatrix} \sum_{i=1}^2 I_{11}(f_i, \sigma_{maxi}, R_i) & \sum_{i=1}^2 I_{11}(f_i, \sigma_{maxi}, R_i) X(f_i) \\ \sum_{i=1}^2 I_{11}(f_i, \sigma_{maxi}, R_i) X(f_i) & \sum_{i=1}^2 I_{11}(f_i, \sigma_{maxi}, R_i) X(f_i)^2 \end{pmatrix}$$

Table 1: Model parameters for material C2 (plain concrete), CF1 (steel-fibre reinforced concrete) and CF2 (polypropylene-fibre reinforced concrete) and fatigue test conditions in [1] and [19].

Mat.	b	c	λ [MPa]	K	σ_{min_0} [MPa/s]	$\dot{\sigma}_0$ [MPa]	σ_{max} [MPa]	R -
C2	0.063	-0.011	76.1	19.8	3.1	0.2	66.4	0.3
CF1	0.049	0.0066	76.1	31.0	12.0	0.2	75.6	0.3
CF2	0.052	-0.001	70.2	15.8	4.8	0.2	62.9	0.3

and its determinant

$$\det(M) = \frac{1}{4} I_{11}(f_1, \sigma_{max1}, R_1) I_{11}(f_2, \sigma_{max2}, R_2) [X(f_1) - X(f_2)]^2. \quad (10)$$

Since I_{11} is non-increasing with respect to both σ_{max} and R , the best of these designs will be supported at the smallest possible values of them. It remains for the function to be maximised solely for the variable f .

For demonstration purposes, we apply the optimal design procedure to the plain concrete C2 and the reinforced concrete CF1 and CF2, referred in [1, 19] and shown in Table 1. First we consider the design spaces $f \in (0, 10]$, $\sigma_{max} \in [0.5\lambda, \lambda]$ and $R \in (0.0001, 1]$, where 0.0001 is a small value chosen to avoid a zero stress ratio.

By maximising Eq. (10), the determinant of the FIM for each concrete, the best two-point designs with equal weights are obtained as follows

$$\xi_{C2}^* = \left\{ \begin{array}{cc} (0.0759, 38.05, 0.0001) & (10, 38.05, 0.0001) \\ 1/2 & 1/2 \end{array} \right\}, \quad (11)$$

$$\xi_{CF1}^* = \left\{ \begin{array}{cc} (0.0339, 38.05, 0.0001) & (10, 38.05, 0.0001) \\ 1/2 & 1/2 \end{array} \right\}, \quad (12)$$

$$\xi_{CF2}^* = \left\{ \begin{array}{cc} (0.0458, 35.10, 0.0001) & (10, 35.10, 0.0001) \\ 1/2 & 1/2 \end{array} \right\}. \quad (13)$$

The sensitivity function Eq. (2) is used to check whether these designs are actually D-optimal or not. In Fig. 1, we plot the sensitivity function and

Figure 1: (For the simplified fatigue model) Sensitivity function for the D-optimal design over the entire frequency range, from 0 to 10 Hz, (left column) and the zoomed view at the top-left corner (right column) for (a) C2, (b) CF1 and (c) CF2, where the maximum is located at 0.0759 Hz, 0.0339 Hz and 0.0458 Hz, respectively.

191 the zoomed view of the top-left corner for C2, CF1 and CF2 respectively. As
 192 mentioned before, once the values of σ_{max} and R were fixed, the maximum
 193 (two for a two-parameter design) is attained at two points of the design; one
 194 corresponds to a low-loading frequency, the other, the maximum frequency
 195 (10 Hz) allowed.

In order to check dependency on the choice of the nominal parameters, a robustness analysis is carried out. Suppose that the values of the real parameters are (b_t, c_t) . The FIM of the corresponding optimal design, ξ_t^* , for them is computed. Consider another design, for example one of those given in Eqs. (11-13), which we denote as ξ_n . The efficiency of the design ξ_n is

measured by

$$Eff_{(b_t, c_t)}(\xi^*) = \left[\frac{\det(M(b_t, c_t; \xi_n))}{\det(M(b_t, c_t; \xi_t^*))} \right]^{\frac{1}{2}}.$$

196 For all values of b and c ranging between 50% and 150% of their initial values,
 197 the efficiency, in all the three cases, surpasses 99%.

Figure 2: (Simplified fatigue model) Sensitivity function for the D-optimal design over the entire frequency range, from 0 to 4 Hz, (left column) and the zoomed view at the top-left corner (right column) for (a) C2, (b) CF1 and (c) CF2, where the maximum is located at 0.1199 Hz, 0.0879 Hz and 0.0880 Hz, respectively.

198 Since, in the experimental design, both the maximum stress, σ_{max} , and
 199 the stress ratio, R , are fixed, as can be seen from Table 1, we apply these
 200 values to carry out the optimisation procedure again. The following designs

(see Fig. 2) are obtained:

$$\xi_{C2}^* = \left\{ \begin{array}{cc} (0.1199, 66.4, 0.3) & (4, 66.4, 0.3) \\ 1/2 & 1/2 \end{array} \right\}, \quad (14)$$

$$\xi_{CF1}^* = \left\{ \begin{array}{cc} (0.0879, 75.6, 0.3) & (4, 75.6, 0.3) \\ 1/2 & 1/2 \end{array} \right\}, \quad (15)$$

$$\xi_{CF2}^* = \left\{ \begin{array}{cc} (0.0880, 62.9, 0.3) & (4, 62.9, 0.3) \\ 1/2 & 1/2 \end{array} \right\}. \quad (16)$$

The experimental designs used in practice is equally replicated at four points for each material, that is,

$$\xi = \left\{ \begin{array}{cccc} (4^{-2}, \sigma_{max}, 0.3) & (4^{-1}, \sigma_{max}, 0.3) & (4^0, \sigma_{max}, 0.3) & (4^1, \sigma_{max}, 0.3) \\ 1/4 & 1/4 & 1/4 & 1/4 \end{array} \right\},$$

where σ_{max} is a constant for each material. The efficiencies of this experimental design, relative to each of the above, are of 87.58%, 86.75% and 86.96% for C2, CF1 and CF2, respectively.

4. Applications to the complete fatigue model

Taking into account the loading frequency, Saucedo *et al.* [1] proposed a new relationship between dynamic and static compressive strengths; namely

$$\sigma_{max_0} = \sigma_{max} \left(\frac{\dot{\sigma}_0}{2f\Delta\sigma} \right)^{0.014 \exp\{\gamma f\}} \equiv \sigma_{max} d^{0.014 \exp\{\gamma f\}}, \quad (17)$$

where the exponent was fitted for frequencies below 10 Hz. The difference between the above equation and Eq. (5) lies in the fact that the hardening coefficient is explicitly dependent on the loading frequency. As a result, a new model parameter, γ , is added. Plugging Eq. (17) into Eq. (3) to determine the cumulative distribution function, which we term as the *complete* fatigue model, we obtain

$$F(N) = 1 - \exp \left\{ -\bar{A}^K N^{BK} \right\},$$

212 where

$$\begin{aligned}\bar{A} &= S_{mx} d^{0.014 \exp\{\gamma f\}} - S_{m0}, \\ B &= [b + c \ln(1 + f)](1 - R),\end{aligned}$$

213 and S_{mx} , S_{m0} and d are defined as in Section 3. Thus, once the parameters
214 of the Weibull distribution are estimated, the distribution of N depends on
215 the variable σ_{max} and R in the fatigue test and the frequency, f , as well as
216 the parameters b , c and γ .

217 In the previous section it was shown that the minimum values of σ_{max}
218 and R had to be included in the support of the D-optimal design. For this
219 reason, we expect the same will happen here again. So we will concentrate
220 on the design for the loading frequency, f .

The log-likelihood function for a specific value of f is

$$LL(N) = K \ln(\bar{A}) + \ln(B) + \ln(K) + (BK - 1) \ln(N) - \bar{A}^K N^{BK},$$

221 Since

$$\begin{aligned}\frac{\partial \bar{A}}{\partial \gamma} &= S_{mx} d^{0.014 \exp\{\gamma f\}} 0.014 \exp\{\gamma f\} f \ln(d) \\ &= 0.014 \exp\{\gamma f\} f \ln(d) (\bar{A} + S_{m0}), \\ \frac{\partial B}{\partial b} &= (1 - R), \\ \frac{\partial B}{\partial c} &= (1 - R) \ln(1 + f) = \frac{\partial B}{\partial b} \ln(1 + f),\end{aligned}$$

222 after differentiating LL twice, we have

$$\begin{aligned}
\frac{\partial^2 LL}{\partial b^2} &= -\frac{(1-R)^2}{B^2} - \bar{A}^K N^{BK} K^2 (1-R)^2 (\ln(N))^2, \\
\frac{\partial^2 LL}{\partial b \partial c} &= \frac{\partial^2 LL}{\partial b^2} \ln(1+f), \\
\frac{\partial^2 LL}{\partial b \partial \gamma} &= -\bar{A}^{K-1} N^{BK} K^2 (1-R) 0.014 \exp\{\gamma f\} f \ln(d) \ln(N) (\bar{A} + S_{m0}), \\
\frac{\partial^2 LL}{\partial c^2} &= \frac{\partial^2 LL}{\partial b^2} [\ln(1+f)]^2, \\
\frac{\partial^2 LL}{\partial c \partial \gamma} &= \frac{\partial^2 LL}{\partial b \partial \gamma} \ln(1+f), \\
\frac{\partial^2 LL}{\partial \gamma^2} &= -\frac{K 0.014 \exp\{\gamma f\} f^2 \ln(d) (\bar{A} + S_{m0})}{\bar{A}^2} \left\{ -\bar{A} + 0.014 \exp\{\gamma f\} \ln(d) S_{m0} N^{2BK} \right. \\
&\quad \left. + \bar{A}^K N^{BK} [\bar{A} - 0.014 \exp\{\gamma f\} \ln(d) S_{m0} + (\bar{A} + S_{m0}) 0.014 \exp\{\gamma f\} \ln(d) K] \right\}.
\end{aligned}$$

Defining

$$H(f) = \frac{0.42278 - K \ln(\bar{A})}{b + c \ln(1+f)}, \quad M(f) = \frac{K 0.014 \exp\{\gamma f\} f \ln(d) (\bar{A} + S_{m0})}{\bar{A}}$$

223 and $X(f) = \log(1+f)$ as before, the elements of the FIM are the following
224 expected values

$$\begin{aligned}
I_{11}(b, c, \gamma; f) &= E \left[-\frac{\partial^2 LL}{\partial b^2} \right] = \frac{(\pi^2/6)}{[b + c \ln(1+f)]^2} + H(f)^2 \equiv I_{11}(f), \\
I_{12}(b, c, \gamma; f) &= E \left[-\frac{\partial^2 LL}{\partial b \partial c} \right] = I_{11}(f) X(f) \equiv I_{12}(f), \\
I_{13}(b, c, \gamma; f) &= E \left[-\frac{\partial^2 LL}{\partial b \partial \gamma} \right] = H(f) M(f) \equiv I_{13}(f), \\
I_{22}(b, c, \gamma; f) &= E \left[-\frac{\partial^2 LL}{\partial c^2} \right] = I_{11}(f) [X(f)]^2 \equiv I_{22}(f), \\
I_{23}(b, c, \gamma; f) &= E \left[-\frac{\partial^2 LL}{\partial c \partial \gamma} \right] = I_{13}(f) X(f) \equiv I_{23}(f), \\
I_{33}(b, c, \gamma; f) &= E \left[-\frac{\partial^2 LL}{\partial \gamma^2} \right] = [M(f)]^2 \frac{(\bar{A} + S_{m0})K - S_{m0} + 2S_{m0}\bar{A}^{-2K}}{(\bar{A} + S_{m0})K} \\
&\equiv I_{33}(f).
\end{aligned}$$

Again E stands for the mathematical expectation; that is the integral of the quantity multiplied by the density function, or likewise the likelihood function. Typical integration methods were used to find these expressions. Consequently, the FIM at a particular value of f can be expressed as

$$I(f) = \begin{pmatrix} I_{11}(f) & I_{11}(f)X(f) & I_{13}(f) \\ I_{11}(f)X(f) & I_{11}(f)X(f)^2 & I_{13}(f)X(f) \\ I_{13}(f) & I_{13}(f)X(f) & I_{33}(f) \end{pmatrix}.$$

For an approximate design with k points in its support,

$$\xi = \begin{Bmatrix} f_1 & f_2 & \cdots & f_k \\ p_1 & p_2 & \cdots & p_k \end{Bmatrix},$$

the FIM is

$$M(b, c, \gamma; \xi) = \begin{pmatrix} \sum_{i=1}^k I_{11}(f_i)p_i & \sum_{i=1}^k I_{11}(f_i)X(f_i)p_i & \sum_{i=1}^k I_{13}(f_i)p_i \\ \sum_{i=1}^k I_{11}(f_i)X(f_i)p_i & \sum_{i=1}^k I_{11}(f_i)X(f_i)^2p_i & \sum_{i=1}^k I_{13}(f_i)X(f_i)p_i \\ \sum_{i=1}^k I_{13}(f_i)p_i & \sum_{i=1}^k I_{13}(f_i)X(f_i)p_i & \sum_{i=1}^k I_{33}(f_i)p_i \end{pmatrix}$$

and its determinant

$$\begin{aligned} \det(M) &= \sum_{i=1}^k I_{33}(f_i)p_i \sum_{i=1}^k I_{11}(f_i)p_i \sum_{j>i}^k I_{11}(f_j)p_j [X(f_j) - X(f_i)]^2 \\ &\quad - \sum_{i=1}^k I_{11}(f_i)p_i \left\{ \sum_{j \neq i}^k I_{13}(f_j)p_j [X(f_j) - X(f_i)] \right\}^2. \end{aligned}$$

As mentioned before, we concentrate on designing for loading frequency. In order to compare with the experimental design presented in [1], the same values for σ_{max} and R are adopted as minimal values in our design. The nominal values assumed for the fatigue tests are listed in Table 2.

After some computations, the designs maximising the determinants of the FIMs for C2, CF1 and CF2 when $f \in (0, 10]$ are as follows

$$\xi_{C2}^* = \begin{Bmatrix} 0.1087 & 10 \\ 1/3 & 2/3 \end{Bmatrix}, \quad (18)$$

Table 2: Summary of the model parameters for the materials considered and fatigue test conditions in [1] and [19].

Mat.	b	c	γ	λ [MPa]	K	σ_{min_0} [MPa/s]	$\dot{\sigma}_0$ [MPa]	σ_{max} [MPa]	R
C2	0.061	0.0105	0.24	76.1	19.8	3.1	0.2	66.4	0.3
CF1	0.049	0.0066	0	76.1	31.0	12.0	0.2	75.6	0.3
CF2	0.052	0.0035	0.086	70.2	15.8	4.8	0.2	62.9	0.3

$$\xi_{CF1}^* = \begin{Bmatrix} 0.1229 & 10 \\ 1/3 & 2/3 \end{Bmatrix}, \quad (19)$$

$$\xi_{CF2}^* = \begin{Bmatrix} 0.1181 & 10 \\ 1/3 & 2/3 \end{Bmatrix}. \quad (20)$$

The sensitivity function is used to check whether these designs are actually D-optimal or not. Figure 3 shows that this function attains the maximum at the two points of the designs for the three concretes, C2, CF1 and CF2. As mentioned above, since the model is nonlinear, the optimal design will depend on the nominal values of the parameters. A sensitivity analysis versus an alternative choice of the parameters is now performed. For this, different possible true values of the parameters, $\theta_t = (b_t, c_t, \gamma_t)$, around the nominal values will be checked. As a first step, the optimal design for those possible true values is computed, say ξ_t^* . Then the efficiency of the designs (18), (19) and (20), obtained for the nominal values, will be computed using the FIM built with θ_t ; namely

$$Eff_{\theta_t}(\xi^*) = \left[\frac{\det(M(\theta_t, \xi^*))}{\det(M(\theta_t, \xi_t^*))} \right]^{\frac{1}{3}}.$$

This efficiency is computed for values of b_t and c_t between 0.5 and 1.5 times the nominal values and γ_t from 0.75 to 1.25 times the nominal value. In all cases the efficiency is over 99%. Since the term $\log(\bar{A})$ appears in the FIM,

Figure 3: (Complete fatigue model) Sensitivity function for the D-optimal design over the entire frequency range from 0 to 10 Hz (left column) and the zoomed view at the top-left corner (right column) for (a) C2, (b) CF1, and (c) CF2, where the maximum is located at 0.1087 Hz, 0.1229 Hz and 0.1181 Hz, respectively.

the values of γ_t are chosen to preserve the positiveness of $\bar{A}$. This is the reason why less variation is considered for the parameter γ_t .

The experimental design used in [1] to estimate the parameters was

$$\xi_e = \left\{ \begin{array}{cccc} 4^{-2} & 4^{-1} & 4^0 & 4^1 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{array} \right\}.$$

and the efficiency of this design for the different concretes is $Eff(\xi_e, \xi_{C2}^*) = 0.64$, $Eff(\xi_e, \xi_{CF1}^*) = 0.6465$ and $Eff(\xi_e, \xi_{CF2}^*) = 0.65$. This means that using the D-optimal design, 35-36% of experiments can be saved while still obtaining the same statistical results in terms of power of hypotheses tests on the parameters.

Figure 4: (Complete fatigue model) Sensitivity function for the D-optimal design over the entire frequency range from 0 to 4 Hz (left column) and the zoomed view (right column) for (a) C2, (b) CF1, and (c) CF2, where the maximum is located at 0.0760 Hz, 0.0879 Hz and 0.0799 Hz, respectively.

246 In the following design, we fix σ_{max} and R as given in Table 2. The
 247 optimisation procedure is carried out for the loading frequency f solely in
 248 the range of $[0, 4]$ Hz. The following results (see Fig. 4) are obtained

$$\xi_{C2}^* = \left\{ \begin{array}{cc} 0.0760 & 4 \\ 1/3 & 2/3 \end{array} \right\}, \quad (21)$$

$$\xi_{CF1}^* = \left\{ \begin{array}{cc} 0.0879 & 4 \\ 1/3 & 2/3 \end{array} \right\}, \quad (22)$$

$$\xi_{CF2}^* = \left\{ \begin{array}{cc} 0.0799 & 4 \\ 1/3 & 2/3 \end{array} \right\}. \quad (23)$$

251 Consequently, the efficiencies of the experimental design carried out in [1]

$$\xi = \left\{ \begin{array}{cccc} (4^{-2}, \sigma_{max}, 0.3) & (4^{-1}, \sigma_{max}, 0.3) & (4^0, \sigma_{max}, 0.3) & (4^1, \sigma_{max}, 0.3) \\ 1/4 & 1/4 & 1/4 & 1/4 \end{array} \right\}$$

252 are 64.01%, 65.27% and 64.82% for C2, CF1 and CF2, respectively.

253 5. Summary and conclusions

254 We have developed a general procedure for obtaining the optimal design
 255 in a fatigue characterisation process and applied the concept of optimal ex-
 256 perimental design to a fatigue model based on the initial Weibull distribution
 257 of the static strength of concrete-like materials. The Fisher Information Ma-
 258 trix (FIM) is obtained both for the simplified and complete fatigue model. In
 259 particular, the optimisation process is carried out for given maximum stress,
 260 stress ratio and loading frequency ranges and a D-optimal design is obtained.
 261 The previously carried out experimental designs is shown to be 87% and 65%
 262 efficient for the simplified and complete fatigue model, respectively. In other
 263 words, 13% or 35% of the experiments could have been saved if the attained
 264 optimal experimental designs had been considered. In addition, a robustness
 265 analysis was carried out to check the dependence of the choice of the nominal
 266 model parameters; the efficiency surpasses 99% when b and c vary between
 267 50% and 150%, and γ changes between 75% and 125% of their respective
 268 nominal values.

269 Even though the procedure has been demonstrated for a chosen fatigue
 270 model, it is applicable to any fatigue model for any other material.

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276 Appendix

σ_{max}	Maximum stress under fatigue loading
σ_{min}	Minimum stress under fatigue loading
σ_{max_0}	Equivalent static value of the dynamic stress, σ_{max}
$\Delta\sigma$	Stress range $\sigma_{max}-\sigma_{min}$
R	Stress ratio defined as $\sigma_{min}/\sigma_{max}$
σ_f	Failure stress
σ_{min_0}	Fatigue endurance limit
$\dot{\sigma}_0$	Loading rate of the quasi-static test
σ_c	Critical stress, can be compressive (f_c), tensile (f_t)
N	Number of cycles to failure (fatigue life)
λ, K	Scale and shape parameter of the Weibull distribution
S_{mx}	Normalised maximum stress σ_{max}/λ
S_{m0}	Normalised minimum stress σ_{min_0}/λ
277 PF	Probability of failure in any point of the domain $\sigma_f - \ln N$
f	Loading frequency employed in a fatigue test
b, c	Parameter that adjusts the relation between $\ln N$ and f, R
γ	Coefficient that takes into consideration of loading frequency for material strength under dynamic loading
ξ	approximate design as a probability measure
x	the vector of covariates defining a particular experiment
β	vector of model parameters
$E[\cdot]$	mathematical expectation
$LL(\beta; t, x)$	the log-likelihood function of the observation t at x
$I(\beta; x)$	Fisher Information Matrix at point x
$M(\beta; \xi)$	Fisher Information Matrix at design ξ

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