RESEARCH ARTICLE

Existence and approximation of nonlocal optimal design problems driven by parabolic equations

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This work is a follow-up to a series of articles by the authors where the same topic for the elliptic case is analyzed. In this article, a class of nonlocal optimal design problem driven by parabolic equations is examined. After a review of results concerning existence and uniqueness for the state equation, a detailed formulation of the nonlocal optimal design is given. The state equation is of nonlocal parabolic type, and the associated cost functional belongs to a broad class of nonlocal integrals. In the first part of the work, a general result on the existence of nonlocal optimal design is proved. The second part is devoted to analyzing the convergence of nonlocal optimal design problems toward the corresponding classical problem of optimal design. After a slight modification of the problem, either on the cost functional or by considering a new set of admissibility, the G-convergence for the state equation and, consequently, the convergence of the nonlocal optimal design problem are proved.

KEYWORDS

approximation of partial differential equations, integral equations, nonlocal parabolic equations, optimal control

MSC CLASSIFICATION

35J20; 49J22; 45A05; 46N20

1 | INTRODUCTION

One of the most meaningful aspects of the theory of optimal control is the characterization of possible minimizers as well as the minimizing sequences. In general, this task is complex, and therefore, the study is usually performed by means of some kind of relaxation or generalized formulation of the original problem. If the convergence of the relaxed problem toward the original problem could be guaranteed, then two important milestones would be realized. On one side, the generalization of the problem would lead us to consider a larger set of admissible elements, and consequently, a greater capacity to reproduce phenomena would be accomplished. This approach, which is by itself essential for the modelization, together with the fact that the set of admissibility is less strict than in the classical setting, would make the problem more tractable. On the other side, if one could assume convergence of the relaxed problem towards the original problem, then the sequences of solutions of the relaxed one could provide valuable information about the minimizing sequences of the original one.

In line with the strategy that we have just described, we define the generalized problem as the nonlocal adaptation of the original one. The problem we formulate is indeed a family of nonlocal parabolic problems depending on a parameter,

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which is usually called horizon. This parameter is the responsible for the intensity of interaction between pairs of states evaluated upon points that can be quite separate. This nonlocality is present both in the state equation and in the cost functional. For each value of the parameter, and for any admissible diffusion coefficient, the existence and uniqueness of solution for the state equation is a well-known fact. In the first part of the work, we investigate whether the optimal control (design) problem is well posed if the control function is the diffusion coefficient and the boundary conditions are Dirichlet homogeneous. In the second part of the work, once the existence of optimal designs for the nonlocal problem has been proved, we intend to explore the convergence towards the original local problem when the horizon tends to zero. In that process, the key stage is to know if the state equation gamma-converges, or G-converges, toward the state equation of the local problem. We will provide a framework for which this convergence is fulfilled. As a by-product of this study, two different proposals of regularization for the local optimal design problem are obtained.

The literature on nonlocal parabolic problems is extensive and has been used to study interesting types of modelization in different fields of applied mathematics.^{1,2} The nonlocality has served to build models in finance,³ fluid dynamics,^{4,5} elasticity,⁶⁻⁸ biology,^{9,10} image processing,^{11,12} or nonconvex variational analysis.¹³⁻¹⁵ In connection with the development of this sort of problems, a collection of works on analysis have participated in an essential way (see, for instance, Andreu-Vaillio et al,¹⁶ Bourgain et al,¹⁷ Brezis,¹⁸ and Ponce^{19,20}). Special attention and profusion in the context of diffusion have received nonlinear problems, especially the case of fractional p -Laplacian (see previous studies^{2,16,21,22}). Also, in what concerns the numerical analysis, there are multiple contributions to facilitate the comprehension of the equations and the efficiency of different numerical schemes.²³⁻²⁹

However, there is not much literature concerning nonlocal optimal design models. We refer, among others, Andrés and Muñoz,^{30,31} Bonder and Spedaletti,³² D'Elia and Gunzburger,^{33,34} and Zhou and Du^{35,36} mainly in the context of elliptic equations. Also, Bonder et al³⁷ is a valuable paper, where an H-convergence study has been carried out.

1.1 | Formulation of the problem

The problem we analyze concerns the optimization of a given functional that is constrained to a specific type of nonlocal parabolic equation. The optimal design problem can be formulated as

$$\min_{(h,u) \in \mathcal{A}} C(h, u), \quad (1)$$

where $C(\cdot, \cdot)$ is the cost functional, and the functional space $\mathcal{A}$, which will be specified later, is characterized by the nonlocal diffusion equation that any admissible pair (h, u) must satisfy

$$(P_\delta) \equiv \begin{cases} u_t(x, t) + L_\delta(u(x, t)) = f(x, t), & \text{in } \Omega, \\ u(x, t) = 0 & \text{in } \Omega_\delta - \Omega, \quad t \in (0, T), \\ u(x, 0) = g(x) & \text{in } \Omega, \end{cases} \quad (2)$$

where

$$L_\delta(u(x, t)) = -2 \int_{B(x, \delta)} \frac{k_\delta(|x' - x|)}{|x' - x|^2} H(x', x) (u(x', t) - u(x, t)) dx' \quad (3)$$

with δ a fixed, positive, and small parameter. In addition, the following conditions hold:

1. Ω is a bounded Lipschitz domain in $\mathbb{R}^N$ and $\Omega_\delta = \Omega \cup \{\cup_{p \in \partial\Omega} B(p, \delta)\}$ where $B(x, r)$ denotes the open ball centered at $x \in \mathbb{R}^N$ and radius $r > 0$.
2. $g \in L^2(\Omega_\delta)$ and $f \in L^2(0, T; L^2(\Omega))$, where

$$L^2(0, T; L^2(\Omega)) = \left\{ f : [0, T] \rightarrow L^2(\Omega) : \int_0^T \|f\|_{L^2(\Omega)}^2 dt < \infty \right\}.$$

3. H is defined as

$$H(x', x) = \frac{h(x) + h(x')}{2}$$

with h in the space

$$\mathcal{H} = \left\{ h : \Omega_\delta \rightarrow [h_{\min}, h_{\max}] \text{ a.e. } x \in \Omega, h = 0 \text{ in } \Omega_\delta - \Omega \text{ and } \int_\Omega h(x) dx = V_0 \right\}.$$

The constants $0 < h_{\min} < h_{\max}$ are given and satisfy the constraints $h_{\min} |\Omega| < V_0 < h_{\max} |\Omega|$.

4. For each $\delta > 0$, the kernel k_δ , satisfies

(a) k_δ is a non-negative radial function such that

$$\text{supp } k_\delta \subset B(0, \delta) \quad (4)$$

and

$$\frac{1}{N} \int_{B(0, \delta)} k_\delta(|s|) ds = 1. \quad (5)$$

(b) The function $K_r : B(0, \delta) - \{0\} \rightarrow (0, +\infty)$ defined by $K_r(z) = \frac{k_\delta(|z|)}{|z|}$ is integrable

$$K_r \in L^1(B(0, \delta)). \quad (6)$$

(c) The function $K_s : B(0, \delta) - \{0\} \rightarrow [0, +\infty)$ defined by $K_s(z) = \frac{k_\delta(|z|)}{|z|^2}$ is singular near the origin in the sense that

$$\lim_{\theta \rightarrow 0^+} \int_{B(0, \delta) - B(0, \theta)} K_s(z) dz = +\infty. \quad (7)$$

In order to understand the problem, we previously need to know the spaces involved. First of all, we define $L_0^2(\Omega_\delta)$ as

$$L_0^2(\Omega_\delta) = \{u : \Omega_\delta \rightarrow \mathbb{R} : u \in L^2(\Omega; \mathbb{R}) \text{ and } u = 0 \text{ in } \Omega_\delta - \Omega\},$$

and the space

$$X = \{v \in L_0^2(\Omega_\delta) : B_h(u, u) < \infty\}$$

where

$$B_h(u, v) = \int_{\Omega_\delta} \int_{\Omega_\delta} H(x', x) \frac{k_\delta(|x' - x|)}{|x' - x|^2} (u(x', t) - u(x, t)) (v(x', t) - v(x, t)) dx' dx. \quad (8)$$

We also define the following subspace of X ,

$$X_0 = \overline{C_{co}^\infty(\Omega_\delta)},$$

where

$$C_{co}^\infty(\Omega_\delta) = \{v \in C_c^\infty(\Omega_\delta) : v = 0 \text{ en } \Omega_\delta - \Omega, \} \subset X.$$

The closure is defined with respect the norm $\|\cdot\|$ given through the quadratic form $B_h(\cdot, \cdot)$:

$$X_0 = \left\{ v \in X : \exists (v_j) \subset C_{co}^\infty(\Omega_\delta) \text{ such that } \lim_j B_h(v_j - v, v_j - v) = 0 \right\}.$$

We notice that, for each h and δ fixed, both X and X_0 are Hilbert spaces with the inner product $B_h(\cdot, \cdot)$. These spaces depends on δ but not on h because the underlying norms are equivalents.³⁸ Another important issue for each δ is the chain of dense and continuous embeddings

$$X_0 \subset L_0^2 \subset X'_0,$$

where X'_0 denotes the dual of X_0 . We are able to identify (a) X_0 with a dense subspace of L_0^2 (as in the classical setting $C_c^\infty(\Omega) \subset H_0^1(\Omega)$ is dense in $L^2(\Omega)$); (b) $L_0^2(\Omega_\delta)$ with itself, by means of its own inner product (as usual for a Hilbert space); (c) $L_0^2(\Omega_\delta)$ with a subspace of X'_0 , so that any function of $L_0^2(\Omega_\delta)$ acts on X_0 via the inner product of $L_0^2(\Omega_\delta)$ (Andrés³⁸). Recall that the above three separable Hilbert spaces constitute what is called a *Gelfand triplet* (see previous studies³⁹⁻⁴¹ and Appendix A).

We finally define

$$Y_0 = L^2(0, T; X_0); \quad (9)$$

that is, the space

$$Y_0 = \left\{ u(\cdot, t) \in X_0 : \int_0^T \|u\|_{X_0}^2 dt < \infty \right\},$$

which is also a Hilbert space. The solution of the problem (P_δ) defined in (2) must be understood in a weak sense:

Definition 1. We say that $u \in Y_0 = L^2(0, T; X_0)$ is a weak solution of Problem (2), if $u_t \in L^2(0, T; X'_0)$, u satisfies the evolution equation of (P_δ) , namely,

$$\langle u_t, v \rangle + B_h(u, v) = (f(x, t), v) \quad (10)$$

for any $v \in X_0$ and a.e. $t \in [0, T]$, and

$$u(x, 0) = g(x), \quad x \in \Omega. \quad (11)$$

Remark 1. The derivative of u with respect to t , u_t , has to be understood in a weak sense. For each $t \in [0, T]$, the solution u has distributional derivative u_t in X'_0 ; here, $(\cdot, \cdot)$ denotes the usual inner product in $L^2(\Omega)$ while $\langle \cdot, \cdot \rangle$ stands for the pairing of X'_0 and X_0 . Henceforth, we will use this short notation, when there is no risk of misunderstanding.

Since $X_0 \subset L^2_0 \subset X'_0$ is a Gelfand triplet, we are allowed to use classical results to ensure that any solution of (2), after an appropriate modification on a null set, belongs to the $C([0, T]; L^2_0(\Omega_\delta))$. Thus, the initial condition (11) makes sense. See Theorem 8 in Appendix A.

Finally, the cost functional in (1) is defined on the admissible pairs according to the formula

$$C(h, u) = \int_0^T \int_{\Omega_\delta} \int_{\Omega_\delta} F(t, x, x', u(x', t), u(x, t)) dx' dx dt,$$

where the integrand $F : (0, T) \times \Omega_\delta \times \Omega_\delta \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to verify the following conditions:

1. $F(\cdot, \cdot, \cdot, u_1, u_2)$ is a measurable function for any $(u_1, u_2) \in \mathbb{R} \times \mathbb{R}$.
2. $F(t, x_1, x_2, \cdot, \cdot)$ is a lower semicontinuous function for any $(t, x_1, x_2) \in (0, T) \times \Omega_\delta \times \Omega_\delta$.

Consequently, the optimal design problem (1)-(2) can be explicitly written as

$$\min_{(h, u) \in \mathcal{A}} \int_0^T \int_{\Omega_\delta} \int_{\Omega_\delta} F(t, x, x', u(x', t), u(x, t)) dx' dx dt \quad (12)$$

where the set of admissibility is

$$\mathcal{A} = \{(h, u_\delta) \in \mathcal{H} \times Y_0 : (h, u_\delta) \text{ satisfy (2)}\}. \quad (13)$$

Definition 2. Fixed $\delta > 0$, we say that $(h_\delta^*, u_\delta^*) \in \mathcal{A}$ is a solution of the optimal design problem (12)-(13) if

$$C(h_\delta^*, u_\delta^*) \leq C(h, u)$$

for any $(h, u) \in \mathcal{A}$.

1.2 | Contributions and organization of the paper

Section 2 is a series of statements indispensable to understand the techniques we have employed. In Section 2.1, the basics about the spectral analysis of nonlocal elliptic operators and the existence of solution for the nonlocal parabolic equations is recalled. Section 2.2 states the convergence of the solution of the nonlocal parabolic problem towards the solution of the corresponding local parabolic equation, when the horizon tends to zero. In a third part of Section 2, we recall one of our key points of these research; Theorem 1 is a continuity result of the spectra with respect to the controls. In addition, and for the sake of clarity, Section 2.4 contains two compactness results that have been essential in our research, and they will be used again in this article (Theorems 2 and 3). All details for the derivation of these achievements can be found at papers.^{30,31,38,42,43}

The main result of this paper is the existence of nonlocal optimal design (Theorem 4). The proof is given in Section 3. Next, Section 4 is devoted to the convergence of the sequence of the optimal designs. The aim is to build some particular nonlocal design problems for which the existence of nonlocal optimal controls is ensured and such that, in the limit, these controls converge to the solution of a local design problem (Theorems 6 and 7). These results previously require a G-convergence result for the nonlocal parabolic equation (Theorem 5).

2 | PRELIMINARIES

In this section, we recall some essential results.

2.1 | Spectral analysis and existence of solution for the nonlocal parabolic equation

The first issue we must underline is the spectral analysis of the nonlocal operators B_h . The main result is the existence of a basis of eigenfunctions $\{w_\delta^{(k)}\} \subset X_0$, with eigenvalues $\gamma_k^{nl}(\delta)$ for the nonlocal operator B_{h_δ} , so that for any $v \in X_0$, we can write

$$B_h(w_\delta^{(k)}, v) = \gamma_k^{nl}(\delta) (w_\delta^{(k)}, v)_{L^2(\Omega) \times L^2(\Omega)}.$$

The sequence of eigenvalues $\gamma_k^{nl}(\delta)$ is nondecreasing and $\gamma_k^{nl}(\delta) \rightarrow +\infty$ when $k \rightarrow +\infty$. Concerning the eigenfunctions, we know $\{w_\delta^{(k)}\}$ is an orthonormal basis in $L_0^2(\Omega_\delta)$ and orthogonal in X_0 . The proof of these claims directly follows from basics tools of linear algebra and makes use of the compactness embedding $X_0 \subset L_0^2(\Omega_\delta)$ (see Theorem 2 below). Consult Andrés,³⁸ Andrés and Muñoz,⁴² and Servadei and Valdinoci⁴⁴ for a detailed analysis.

The described framework and the above spectral results provide the appropriate tool to solve the problem (2): the Galerkin-Fourier method. We know that for a given $h \in \mathcal{H}$ and a $\delta > 0$ fixed, there exists a unique solution $u_\delta \in Y_0$ of Equation (2). Moreover, if $f(x, t) = \sum_{k=1}^{\infty} f_{\delta k}(t) w_\delta^{(k)}(x)$, where $(f_{\delta k})_k$ is the sequence of Fourier coefficients given by $f_{\delta k}(t) = \int_{\Omega} f(x, t) w_\delta^{(k)}(x) dx$, and $(g_{\delta k})_k$ is the sequence of Fourier coefficients of g , then

$$u_\delta(x) = \sum_{k=1}^{\infty} d_\delta^{(k)}(t) w_\delta^{(k)}(x), \quad (14)$$

where $d_\delta^{(k)}$ is the solution to the initial value problem

$$\begin{cases} (d_\delta^{(k)}(t))' + \gamma_k^{nl}(\delta) d_\delta^{(k)}(t) - f_{\delta k}(t) = 0, \\ d_\delta^{(k)}(0) = g_{\delta k}. \end{cases} \quad (15)$$

Again, the details of all these results can be consulted in Andrés,³⁸ Andrés and Muñoz,⁴² and Servadei and Valdinoci.⁴⁴

2.2 | Convergence to the local problem

Next, a relevant aspect that we must have in mind is the asymptotic behavior of the sequence of solutions $(u_\delta)_\delta$ when $\delta \rightarrow 0$ with $h \in \mathcal{H}$. Consequently, some notes related to the local problem are in order. The classical or local problem is formulated as follows: Find $u \in W(0, T)$ such that

$$(P) \equiv \begin{cases} w_t(x, t) - \operatorname{div}(h(x)w(x, t)) = f(x, t), \text{ in } \Omega, \\ w(x, t) = 0 \text{ in } \partial\Omega, \\ w(x, 0) = g(x). \end{cases} \quad (16)$$

The space $W(0, T)$ is a Hilbert space described as

$$W(0, T) = \{v \in L^2(0, T; H_0^1(\Omega)) : v_t \in L^2(0, T; H^{-1}(\Omega))\}$$

and whose inner product is given by

$$(u, v) = \int_0^T (\nabla_x u(x, t), \nabla_x v(x, t))_{L^2(\Omega)} dt + \int_0^T \langle u_t(x, t), v_t(x, t) \rangle_{H^{-1}(\Omega)} dt.$$

In this framework, it is known the following result of convergence: From the sequence of solutions of (2), $(u_\delta)_\delta$, we can extract a subsequence that converges strongly in Y_0 to a function $u \in W(0, T)$ when $\delta \rightarrow 0$. Moreover, u is the weak solution of the local problem (16). We refer the reader to previous works,^{38,43,45,46} for instance, for the details concerning this kind of nonlocal convergence.

2.3 | Convergence of the spectra

The intrinsic structure of the optimal design problem we have just described lies also in the convergence of the spectra. In this case, the asymptotic is not with respect to the horizon parameter δ , it has to do with the limit when we consider a minimizing sequence.

Let $(h_j)_j$ be a sequence of admissible designs. From this sequence, we extract a subsequence whose weak-* limit in $L^\infty(\Omega_\delta)$ is h . Then, with t and δ fixed, and for each j , as we have already indicated, the operator B_{h_j} has a unitary basis of eigenfunctions. The notation is $\gamma_1^{nl}(j)$ is the first eigenvalue and $w_j^{(1)}$ its corresponding eigenfunction. For $k \geq 2$, the eigenvalue is $\gamma_k^{nl}(j)$ and its corresponding set of eigenfunctions $\cup_{i=1}^{n_k} w_{ij}^{(k)}$ ($n_k \in \mathbb{N}$). In the same way for B_h , we assume its first eigenfunction is $\psi^{(1)}$ with eigenvalue γ_1^{nl} ; and for $k \geq 2$, the set $\cup_{i=1}^{l_k} \psi_i^{(k)}$ will denote the set of eigenvector whose eigenvalue is γ_k^{nl} .

Under these circumstances, we have

Theorem 1 (Andrés and Muñoz³⁰).

1. Concerning the first eigenvalue, $\gamma_1^{nl}(j)$, and the first eigenfunction $w_j^{(1)}$ ($l_1 = 1$), there is a subsequence of indexes j still denoted by j such that $\lim_j \gamma_1^{nl}(j) = \gamma_1$ and $\lim_j w_j^{(1)} = \psi^{(1)}$ in $L^2(\Omega_\delta)$.
2. For each $k \geq 2$, we have $n_k = l_k$, and there is a subsequence of indexes j , still denoted by j , such that $\lim_j \gamma_k^{nl}(j) = \gamma_k$ and a set of functions $\cup_{i=1}^{n_k} w_{ij}^{(k)}$ in X_0 such that $\lim_j w_{ij}^{(k)} = w_i^{(k)}$ strong in L^2 , where $w_i^{(k)}$ are eigenfunctions for the nonlocal problem defined by B_h with eigenvalues γ_k , for $i = 1, \dots, n_k$. Moreover $\mathcal{L} \left\{ \cup_{i=1}^{n_k} w_i^{(k)} \right\} = \mathcal{L} \left\{ \cup_{i=1}^{n_k} \psi_i^{(k)} \right\}$ for any k .

2.4 | Compactness results

The proof of Theorem 1 straightforwardly follows from some essential results by Brezis, Ponce and other authors. In particular, it makes use of these two interesting compactness tools:

Theorem 2. For each $\delta > 0$ and $h \in \mathcal{H}$ fixed.

1. (Andreu-Vaillou et al,¹⁶ Ponce,¹⁹ Andrés and Muñoz⁴²) There is a positive constant A_p , depending only on Ω such that for every $\psi \in X_0$,

$$B_h(\psi, \psi) \geq A_p \|\psi\|_{L^2(\Omega)}^2 \quad (17)$$

(nonlocal Poincaré inequality).

2. (Andrés and Muñoz^{31,42}) For any given constant $C > 0$, the set

$$\mathcal{X}_C(\delta) = \left\{ \psi \in L_0^2(\Omega_\delta) : B_h(\psi, \psi) < C \right\} \quad (18)$$

is compactly imbedded in $L_0^2(\Omega_\delta)$.

Theorem 3 (Bourgain et al,¹⁷ Brezis,¹⁸ Ponce,¹⁹ and Andrés and Muñoz⁴²). If $(\psi_\delta)_\delta$ is a bounded sequence in $L_0^2(\Omega_\delta)$ and there is a positive constant C such that $B_h(\psi_\delta, \psi_\delta) \leq C$, for any $\delta \leq \delta_0$, then $(\psi_\delta)_\delta$ is relatively compact in $L_0^2(\Omega_\delta)$. Moreover, if $(\psi_{\delta_j})_j$ is a subsequence such that $\psi_{\delta_j} \rightarrow \psi$ strong in $L^2(\Omega)$, if $j \rightarrow +\infty$, then $\psi \in H^1(\Omega)$.

3 | EXISTENCE OF OPTIMAL DESIGNS

The aim is to proof the following result:

Theorem 4. There exists at least a pair $(h, u) \in \mathcal{H} \times Y_0$ solution to the problem (12)-(13).

We base the idea of the proof on some key asymptotic results.

3.1 | Step 1

Assume the parameter $\delta > 0$ is fixed and let $(h_j, u_j)_j$ be a minimizing sequence of the problem (12)-(13). Then

$$\langle (u_j)_t, u_j \rangle + B_{h_j}(u_j, u_j) = (f, u_j)_{L^2(\Omega) \times L^2(\Omega)}.$$

As in the local case, the use of the above variational equation, Theorem 8, and Poincaré inequality (17) yield

$$\begin{aligned} A_p \int_0^T \|u_j\|_{L^2(\Omega_\delta)}^2(t) dt &\leq \int_0^T B_{h_j}(u_j, u_j) dt \\ &\leq \frac{1}{2} \|g\|_{L^2(\Omega)}^2 + \left(\int_0^T \|f\|_{L^2(\Omega)}^2 dt \right)^{1/2} \left(\int_0^T \|u_j\|_{L^2(\Omega)}^2 dt \right)^{1/2}. \end{aligned}$$

This clearly implies that $\int_0^T \|u_j\|_{L^2(\Omega_\delta)}^2(t) dt$ is bounded, and consequently, $\int_0^T B_{h_j}(u_j, u_j) dt$ too.

Now, we shall prove uniform estimations:

Proposition 1. *The sequences $\|u_j\|_{L^2(\Omega_\delta)}^2$ and $B_{h_j}(u_j, u_j)$ are uniformly bounded in j , for a.e. $t \in (0, T]$.*

Proof. By Poincaré inequality (17), it suffices to prove

$$B_{h_j}(u_j, u_j) < C, \quad (19)$$

for any j . Thanks to (14)-(15), we can write

$$u_j(x, t) = \sum_{k=1}^{\infty} d_j^{(k)}(t) w_j^{(k)}(x),$$

where

$$d_j^{(k)}(t) = \int_0^t f_j^{(k)}(s) \exp\left(\gamma_{jk}^{nl}(s-t)\right) ds + g_j^{(k)} \exp\left(-\gamma_{jk}^{nl}t\right).$$

By using the above expansion of u_j and Hölder inequality, we get

$$\begin{aligned} B_{h_j}(u_j(x, t), u_j(x, t)) &= \sum_{k=1}^{\infty} \left(d_j^{(k)}(t)\right)^2 \gamma_{jk}^{nl} \\ &\leq \sum_{k=1}^{\infty} \int_0^t \left(f_j^{(k)}(s)\right)^2 ds \left(2\gamma_{jk}^{nl} \int_0^t \exp\left(2\gamma_{jk}^{nl}(s-t)\right) ds\right) \\ &\quad + \sum_{k=1}^{\infty} \left(g_j^{(k)}\right)^2 2\gamma_{jk}^{nl} \exp\left(-2\gamma_{jk}^{nl}t\right) \\ &\leq \sum_{k=1}^{\infty} \int_0^T \left(f_j^{(k)}(s)\right)^2 ds + \frac{1}{t} \sum_{k=1}^{\infty} \left(g_j^{(k)}\right)^2, \end{aligned}$$

whence we obviously deduce (19). $\square$

In view of the above discussion, Theorem 2 ensures the sequence $(u_j)_j$ is precompact in $L^2(\Omega_\delta)$: Hence, there is $u(\cdot, t) \in L^2(\Omega_\delta)$ such that for a subsequence of $(u_j)_j$,

$$u_j \rightarrow u \text{ strongly in } L^2(\Omega_\delta).$$

Also, since

$$\int_0^T B_{h_j}(u_j, u_j) dt = \int_0^T \|u_j\|_{X_0}^2 dt \leq C,$$

then

$$u_j \rightharpoonup u \text{ weakly in } Y_0.$$

3.2 | Step 2

We are going to see the pair (h, u) , where u is the limit of the sequence $(u_j)_j$ described above and h is the weak-* limit in L^∞ of the sequence $(h_j)_j$, is admissible. Since $u \in Y_0$ and $h \in \mathcal{H}$, it only remains to see whether they fulfill the nonlocal state equation.

By Theorem 1, we know

$$\lim_{j \rightarrow +\infty} \gamma_k^{nl}(j) = \gamma_k^{nl},$$

and

$$\lim_{j \rightarrow +\infty} w_{ij}^{(k)}(x) = w_i^{(k)}(x) \text{ strong in } L^2, \text{ for } i = 1, \dots, n_k,$$

where $\gamma_k^{nl}(j)$ are the eigenvalues of the eigenfunctions $w_{ij}^{(k)}(x)$ for $i = 1, \dots, n_k$, of the nonlocal operator B_{h_j} . In the same way, γ_k^{nl} are the eigenvalues that correspond to the eigenfunctions $w_i^{(k)}(x)$, for $i = 1, \dots, n_k$, of the nonlocal operator B_h . We also recall $\cup_{k=1}^\infty \cup_{i=1}^{n_k} w_{ij}^{(k)}(x)$ and $\cup_{k=1}^\infty \cup_{i=1}^{n_k} w_i^{(k)}(x)$ are orthogonal basis in X_0 . From these assertions, we easily deduce

$$\begin{aligned} \lim_{j \rightarrow +\infty} f_{ij}^{(k)}(t) &= \lim_{j \rightarrow +\infty} (f, w_{ij}^{(k)}) = (f, w_i^{(k)}) = f_i^{(k)}, \\ \lim_{j \rightarrow +\infty} g_{ij}^{(k)} &= \lim_{j \rightarrow +\infty} (g, w_{ij}^{(k)}) = (g, w_i^{(k)}) = g_i^{(k)} \end{aligned}$$

for $i = 1, \dots, n_k$. Thanks to the above limits, we arrive at

$$\lim_{j \rightarrow \infty} u_{ij}^{(k)} = \lim_{j \rightarrow \infty} (u_j, w_{ij}^{(k)}(x)) = (u, w_i^{(k)}) = u_i^{(k)}$$

for $i = 1, \dots, n_k$, and thus,

$$\begin{aligned} u_i^{(k)} &= \lim_{j \rightarrow \infty} u_{ij}^{(k)} = \lim_{j \rightarrow \infty} d_{ij}^{(k)}(t) \\ &= \lim_{j \rightarrow \infty} \int_0^t f_{ij}^{(k)}(s) \exp(\gamma_k^{nl}(j)(s-t)) ds + g_{ij}^{(k)} \exp(-\gamma_k^{nl}(j)t) \\ &= \int_0^t f_i^{(k)}(s) \exp(\gamma_k^{nl}(s-t)) ds + g_i^{(k)} \exp(-\gamma_k^{nl}t) = d_i^{(k)}(t) \end{aligned}$$

for $i = 1, \dots, n_k$, where $f_i^{(k)}(t) = \int_\Omega f(x, t) w_i^{(k)}(x) dx$ and $g_i^{(k)} = \int_\Omega g(x) w_i^{(k)}(x) dx$. If we omit the multiplicity in the notation for the eigenfunctions, we expand the limit u in the basis $\cup_{k=1}^\infty \cup_{i=1}^{n_k} w_i^{(k)}(x)$, and we use the above results of convergence, then we can write

$$\begin{aligned} u(x, t) &= \sum_{k=1}^\infty d_k(t) w^{(k)}(x) \\ &= \sum_{k=1}^\infty \left(\int_0^t f^{(k)}(s) \exp(\gamma_k^{nl}(s-t)) ds + g^{(k)} \exp(-\gamma_k^{nl}t) \right) w^{(k)}(x). \end{aligned} \tag{20}$$

Moreover, this function u is the solution of (10) (see Lions⁴⁷ or Evans⁴⁸).

As a consequence of the above process, we state the convergence

$$\lim_{j \rightarrow \infty} B_{h_j}(u_j, u_j) = B_h(u, u).$$

3.3 | Step 3

We are going to prove the pair (h, u) is a solution of (12)-(13). This admissible pair has been obtained as the limit of the minimizing sequence (h_j, u_j) in the sense that h is the weak-* limit in L^∞ of $(h_j)_j$, and u is both the weak limit in Y_0 and

the strong limit in L^2 of the sequence $(u_j)_j$. Then, we can apply Fatou lemma, if it is necessary for a subsequence, to get

$$\begin{aligned} & \lim_j \int_0^T \int_{\Omega_\delta} \int_{\Omega_\delta} F(t, x, x', u_j(x', t), u_j(x, t)) dx' dx dt \\ & \geq \int_0^T \int_{\Omega_\delta} \int_{\Omega_\delta} F(t, x, x', u(x', t), u(x, t)) dx' dx dt, \end{aligned}$$

which serves to ensure the pair (h, u) is a solution of the local design problem.

4 | CONVERGENCE OF A CLASS OF NONLOCAL OPTIMAL DESIGN PROBLEMS

The motivation is the extension to the parabolic case of some results from Andrés and Muñoz.³¹ The goal is to produce some results that could help us to approximate the solutions of the following local problem (D_0) :

$$\min_{(h,u) \in \mathcal{A}} \int_0^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt \quad (21)$$

where

$$\mathcal{A}_{loc} = \{(h, u) \in \mathcal{H} \times W(0, T) : (h, u) \text{ satisfy (16)}\}. \quad (22)$$

The idea is to work with a nonlocal optimal design problem (D_δ) for which we know existence of optimal designs (h_δ, u_δ) (h_δ the design and u_δ its associated state) and then pass to the limit in order to recover a local problem that could be interpreted as a regularization of the problem (D_{loc}) . We give a more precise explanation of that.

4.1 | Description of the problem

4.1.1 | The nonlocal model

The aim is to analyze the asymptotic when $\delta \rightarrow 0$ of some kind of optimal design problems. The first type of problem we are going to discuss is (D_δ) ; it is defined as follows:

$$\min_{(h,u) \in \mathcal{A}} J_\delta, \quad (23)$$

where the cost functional is

$$J_\delta(h, u) = \int_0^T B(u, u) dt + c B_0(h, h) \quad (24)$$

with

$$B_0(h, h) = \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} k_{\delta_0}(|x' - x|) \frac{|h(x') - h(x)|^2}{|x' - x|^2} dx' dx, \quad (25)$$

and $\delta_0 > 0$ and $c \geq 0$ are given constants. The set of admissibility is

$$\mathcal{A} = \{(h, u) \in \mathcal{H} \times Y_0 : (h, u) \text{ satisfy (2)}\}, \quad (26)$$

and δ_0 and c are given positive numbers.

We shall address another type of problems. They are quite similar to the above class. The only difference hinges on the role played by the nonlocal term B_0 . We define the design problem (D_δ^R) as follows:

$$\min_{(h,u) \in \mathcal{A}^R} J_\delta^R, \quad (27)$$

where the cost functional is

$$J_\delta^R(h, u) = \int_0^T B(u, u) dt, \quad (28)$$

where F is as in Section 1 and the set of admissibility is

$$\mathcal{A}^R = \{(h, u) \in \mathcal{H}_R(\delta_0) \times Y_0 : (h, u) \text{ satisfy (2)}\}, \quad (29)$$

where

$$\mathcal{H}_R(\delta_0) = \{h \in \mathcal{H} : B_0(h, h) < R\}. \quad (30)$$

4.1.2 | The classical model

We describe the local model we are going to solve. It is indeed the corresponding regularization of (21)-(22) that will be solved by means of the minimizers of the corresponding nonlocal optimal design problem, (D_δ) or (D_δ^R) .

The formulation is quite similar and the one that corresponds to the first type (D_δ) is

$$(D_{loc}) \equiv \min_{(h, u) \in \mathcal{A}_{loc}} J_{loc}(h, u) \quad (31)$$

where

$$J_{loc}(h, u) = \int_0^T \int_\Omega |\nabla u(x, t)|^2 dx dt + c B_0(h, h) \quad (32)$$

$$\mathcal{A}_{loc} = \{(h, u) \in \mathcal{H} \times W(0, T) : (h, u) \text{ satisfy (16)}\}. \quad (33)$$

We underline that the only difference between (D_{loc}) and (D_0) is the perturbation implemented in cost by means of the nonlocal term B_0 .

The local version of (D_δ^R) is denoted by (D_{loc}^R) . The cost functional of (D_{loc}^R) is denoted by J_{loc}^R , and it is defined as in (31) but with $c = 0$. The set of admissibility of (D_{loc}^R) is

$$\mathcal{A}_{loc}^R = \{(h, u) \in \mathcal{H}_R(\delta_0) \times W(0, T) : (h, u) \text{ satisfy (16)}\}. \quad (34)$$

Now, the approach to (D_0) is performed by modifying the set of admissibility. The only distinction between (D_0) and (D_{loc}^R) is the nonlocal constraint $B_0(h, h) < R$.

4.1.3 | Concerning the bound on the controls

It is true that the assumption made about the controls inevitably implies a certain regularity for them. If we try to solve the original problem (21)-(22), then, for a fixed small δ , we should have to deal with an approximation of the solution for the nonlocal version, a sequence $(h_{\delta j}, u_{\delta j})$. Under these circumstances, it would be interesting to analyze the evolution of the bound on the controls in terms of j and δ : Compute $B_0(h_{\delta j}, h_{\delta j})$ and allocate a certain positive constant ϱ , which depends on j and δ , such that

$$B_0(h_{\delta j}, h_{\delta j}) = \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} k_{\delta_0}(|x' - x|) \frac{|h_{\delta j}(x') - h_{\delta j}(x)|^2}{|x' - x|^2} dx' dx \leq \varrho(j, \delta).$$

The key point in order to contribute to solving the original problem (21)-(22) is the analysis of the sequence ϱ if we let $j \rightarrow +\infty$ with δ fixed and small. This could be a particular way to provide any sort of information concerning the behavior of the optima or about the complexity of the minimization sequences.

It is clear that ϱ is expected to increase if the fineness of the partition goes to zero, that is, if $j \rightarrow +\infty$. However, if we fix δ and then j , and for any given minimizing sequence of pairs $(h_{\delta j}, u_{\delta j})$, we eventually compute the corresponding ϱ , then the regularized problem (27)-(30), defined by means of $\mathcal{H}_\varrho(\delta_0)$, is a finer approximation. Indeed, the solution of the latter problem (h_j^*, u_j^*) is optimal because necessarily $h_{\delta j} \in \mathcal{H}_\varrho(\delta_0)$, and thus, the cost achieved is below; that is,

$$J_\delta(h_j^*, u_j^*) \leq J_\delta(h_{\delta j}, u_{\delta j}).$$

Another ingredient we have to note, inside B_0 , is the singular term $\frac{|h(x') - h(x)|^2}{|x' - x|^2}$, which, somehow, can be compensated by the kernel k_{δ_0} if the selected δ_0 is large enough. This factually allows us to consider wider sets of admissible controls,

even controls with large discontinuities. Unfortunately, there is no way to pass to the limit because the assumption on the kernel made in (7).

4.2 | G-convergence

We consider a sequence $(h_\delta)_\delta \subset \mathcal{H}$ and h , its weak-* limit in $L^\infty(\Omega)$ as well. Let $u \in W(0, T)$ be the underlying local state of h , that is, the only solution of the state equation (16):

$$\begin{cases} w_t - \operatorname{div}(h \nabla w) = f, & w \in W(0, T), \\ w(x, t) = 0 & \text{in } \partial\Omega, \\ w(x, 0) = g(x). \end{cases}$$

We can also compute the sequence of solutions $(u_\delta)_\delta$ of the nonlocal state equation (2). Then

$$B_{h_\delta}(u_\delta, w) = (f, w) - \langle (u_\delta)_t, w \rangle \text{ for any } w \in X_0. \quad (35)$$

If we use the notation

$$b_h(u, v) = \int_{\Omega} h(x) \nabla u(x) \cdot \nabla v(x) dx,$$

then it is clear that

$$b_h(u, u) = (f, u) - \langle u_t, u \rangle.$$

After some easy manipulations, we obtain

$$\begin{aligned} b_h(u, u) &= (f, u) - \langle u_t, u \rangle \\ &= (2B_{h_\delta}(u_\delta, u) + 2\langle (u_\delta)_t, u \rangle) - ((f, u) + 2\langle u_t, u \rangle - \langle u_t, u \rangle) \\ &= 2B_{h_\delta}(u_\delta, u) + 2\langle (u_\delta)_t, u \rangle - b_h(u, u) - 2\langle u_t, u \rangle \\ &= (2B_{h_\delta}(u_\delta, u) - b_h(u, u)) - 2\langle u_t - (u_\delta)_t, u \rangle. \end{aligned}$$

Consequently, we can write

$$\begin{aligned} &\lim_{\delta \rightarrow 0} [(B_{h_\delta}(u_\delta, u_\delta) - b_h(u, u)) - 2\langle u_t - (u_\delta)_t, u \rangle] \\ &= \lim_{\delta \rightarrow 0} B_{h_\delta}(u_\delta - u, u_\delta - u) + \lim_{\delta \rightarrow 0} (b_h(u, u) - B_h(u, u)) \\ &\quad + \lim_{\delta \rightarrow 0} (B_h(u, u) - B_{h_\delta}(u, u)). \end{aligned}$$

In addition, we know

$$\lim_{\delta \rightarrow 0} B_h(u, u) = b_h(u, u)$$

and

$$\lim_{\delta \rightarrow 0} B_{h-h_\delta}(u, u) = 0$$

(see Bourgain et al¹⁷ and Andrés and Muñoz^{30, p307}), and therefore,

$$\lim_{\delta \rightarrow 0} (B_{h_\delta}(u_\delta, u_\delta) + 2\langle (u_\delta)_t, u \rangle) \geq b_h(u, u) + 2\langle u_t, u \rangle. \quad (36)$$

We observe that u_δ is the solution of the problem

$$\min_{g \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(g, g) - (f, g) + \langle (u_\delta)_t, g \rangle \right\} \quad (37)$$

and analogously, u is the solution of the problem

$$\min_{g \in W} \left\{ \frac{1}{2} b_h(g, g) - (f, g) + \langle (u)_t, g \rangle \right\}. \quad (38)$$

Thus, thanks to (36), (37), and (38), we obtain the following inequality:

$$\begin{aligned}
& \liminf_{\delta \rightarrow 0} \min_{g \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(g, g) - (f, g) + \langle (u_\delta)_t, g \rangle \right\} \\
&= \lim_{\delta \rightarrow 0} \frac{1}{2} B_{h_\delta}(u_\delta, u_\delta) - (f, u_\delta) + \langle (u_\delta)_t, u_\delta \rangle \\
&= \lim_{\delta \rightarrow 0} \left(-\frac{1}{2} B_{h_\delta}(u_\delta, u_\delta) \right) \\
&= \lim_{\delta \rightarrow 0} \left(-\frac{1}{2} B_{h_\delta}(u_\delta, u_\delta) - \langle (u_\delta)_t, u \rangle + \langle (u_\delta)_t, u \rangle \right) \\
&\leq \lim_{\delta \rightarrow 0} \left[-\frac{1}{2} b_h(u, u) - \langle u_t, u \rangle + \langle (u_\delta)_t, u \rangle \right] \\
&= \lim_{\delta \rightarrow 0} \left[\frac{1}{2} b_h(u, u) - (f, u) + \langle u_t, u \rangle \right] + \\
&\quad + \lim_{\delta \rightarrow 0} \langle (u_\delta)_t - u_t, u \rangle \\
&= \min_{g \in W} \left\{ \frac{1}{2} b_h(g, g) - (f, g) + \langle (u)_t, g \rangle \right\} + \lim_{\delta \rightarrow 0} \langle (u_\delta)_t - u_t, u \rangle.
\end{aligned}$$

As far as the inverse inequality is concerned, it is essential to observe that the same study performed for the existence (Subsection 3.1) applies now: Since u_δ solves the variational formulation, then, for every δ ,

$$\langle (u_\delta)_t, u_\delta \rangle + B_{h_\delta}(u_\delta, u_\delta) = (f, u_\delta)_{L^2(\Omega) \times L^2(\Omega)}$$

holds. Hence, there is positive constant C such that

$$B_{h_\delta}(u_\delta, u_\delta) \leq C$$

for a.e. t . Thus, Theorem 3 implies $u_\delta \rightarrow v \in H_0^1(\Omega)$ strongly in $L^2(\Omega)$. Under these circumstances, we have the

Proposition 2. *If $h_\delta \rightarrow h$ a.e., then*

$$\lim_{\delta \rightarrow 0} B_{h_\delta}(u_\delta, u_\delta) \geq b_h(v, v).$$

Proof. If we define the sequence of measures

$$\mu_\delta(A) = \int_A \left(\int_{B(x, \delta)} k_\delta(|x' - x|) \frac{(u_\delta(x') - u_\delta(x))^2}{|x' - x|^2} dx' \right) dx$$

for any measurable set $A \subset \Omega$, then we clearly have

$$B_{h_\delta}(u_\delta, u_\delta) = \int_\Omega h_\delta(x) d\mu_\delta(x).$$

Moreover,

$$\liminf_{\delta \rightarrow 0} \mu_\delta(A) \geq \liminf_{\delta \rightarrow 0} \int_A \int_A k_\delta(|x' - x|) \frac{(u_\delta(x') - u_\delta(x))^2}{|x' - x|^2} dx' dx \geq \int_A |\nabla u(x)|^2 dx,$$

where the last inequality is due to a generalization of Ponce inequality (see Ponce^{19, p12} and Andrés and Muñoz^{31, Appendix A}). After this, it is enough to use a version of Fatou lemma for measurable sets to conclude. See the details of the proof in Andrés and Muñoz.^{31, p41, Proposition 3.2} $\square$

By applying the above proposition, we get

$$\begin{aligned}
 & \liminf_{\delta \rightarrow 0} \min_{g \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(g, g) - (f, g) + \langle (u_\delta)_t, g \rangle \right\} \\
 &= \lim_{\delta \rightarrow 0} \frac{1}{2} B_{h_\delta}(u_\delta, u_\delta) - \lim_{\delta \rightarrow 0} (f, u_\delta) + \langle (u_\delta)_t, u_\delta \rangle \\
 &\geq \frac{1}{2} b_h(v, v) - (f, v) + \langle u_t, v \rangle + \langle v_t, v \rangle - \langle u_t, v \rangle \\
 &\geq \min_{g \in W} \left\{ \frac{1}{2} b_h(g, g) - (f, g) + \langle u_t, g \rangle \right\} + \langle v_t - u_t, v \rangle.
 \end{aligned}$$

Then, by gathering the above estimations, we arrive at

$$\begin{aligned}
 & \min_{g \in W} \left\{ \frac{1}{2} b_h(g, g) - (f, g) + \langle u_t, g \rangle \right\} + \langle v_t - u_t, v \rangle \\
 &\leq \liminf_{\delta \rightarrow 0} \min_{g \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(g, g) - (f, g) + \langle (u_\delta)_t, g \rangle \right\} \\
 &\leq \min_{g \in W} \left\{ \frac{1}{2} b_h(g, g) - (f, g) + \langle (u)_t, g \rangle \right\} + \lim_{\delta \rightarrow 0} \langle (u_\delta)_t - u_t, u \rangle \\
 &= \min_{g \in W} \left\{ \frac{1}{2} b_h(g, g) - (f, g) + \langle (u)_t, g \rangle \right\} + \langle v_t - u_t, u \rangle.
 \end{aligned} \tag{39}$$

Then

$$\langle v_t - u_t, v - u \rangle \leq 0,$$

and so, for any $s \in (0, T)$,

$$\int_0^s \frac{1}{2} \frac{d}{dt} (\|v(t) - u(t)\|_{L^2}^2) dt \leq 0.$$

Thus,

$$\|v(s) - u(s)\|_{L^2}^2 \leq 0,$$

which amounts to say $u = v$ a.e. Moreover, from (39), we get

$$\lim_{\delta \rightarrow 0} B_{h_\delta}(u_\delta, u_\delta) = b_h(u, u).$$

By taking into account that

$$\begin{aligned}
 \lim_{\delta \rightarrow 0} B_{h_\delta}(u_\delta, u) &= \lim_{\delta \rightarrow 0} ((f, u) - \langle (u_\delta)_t, u \rangle) \\
 &= (f, u) - \langle u_t, u \rangle \\
 &= b_h(u, u),
 \end{aligned}$$

it is straightforward to check the following strong convergence in X_0 :

$$\lim_{\delta \rightarrow 0} B_{h_\delta}(u_\delta - u, u_\delta - u) = 0.$$

The above limit and the dominated convergence theorem provide strong convergence of $(u_\delta)_\delta$ (at least for a subsequence) to u in Y_0 , in the sense that

$$\lim_{\delta \rightarrow 0} \|u_\delta - u\|_{Y_0} = 0.$$

We have proved the following result:

Theorem 5. *If $(h_\delta)_\delta \subset \mathcal{H}$ and $h_\delta \rightarrow h$ a.e. $x \in \Omega$, then*

$$\begin{aligned}
 & \liminf_{\delta \rightarrow 0} \min_{g \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(g, g) - (f, g) + \langle (u_\delta)_t, g \rangle \right\} \\
 &= \min_{g \in W} \left\{ \frac{1}{2} b_h(g, g) - (f, g) + \langle (u)_t, g \rangle \right\}
 \end{aligned}$$

and $u_\delta \rightarrow u$ strongly both in L^2 and also in Y_0 in the sense described above.

4.3 | The local optimal design problem as the limit when $\delta \rightarrow 0$

We analyze the limit of the solutions of the problem (D_δ) . Let (h_δ, u_δ) be the sequence of solutions of (D_δ) . We know $h_\delta \rightharpoonup h$ weak-* in L^∞ .

We notice

$$J_\delta(h_\delta, u_\delta) \leq J_\delta(h^*, u_\delta^*)$$

for any regular admissible $h^* \in \mathcal{H}$ whose associated state is u_δ^* . Then $u_\delta^* \rightarrow u^* \in H^1(\Omega)$ so that

$$\lim_{\delta \rightarrow 0} B_{h^*}(u_\delta^*, u_\delta^*) = b_{h^*}(u^*, u^*);$$

this fact implies $B(u_\delta^*, u_\delta^*)$ is uniformly bounded. By noting that

$$J_\delta(h^*, u_\delta^*) = \int_0^T B(u_\delta^*, u_\delta^*) dt + cB_0(h^*, h^*)$$

and taking into account that $B_0(h^*, h^*)$ is finite, we deduce $J_\delta(h_\delta, u_\delta)$ is uniformly bounded. This fact, in particular, ensures $B_0(h_\delta, h_\delta)$ is uniformly bounded too. That means the sequence $(h_\delta)_\delta$ is included in the set $\mathcal{H}_R(\delta_0)$ for some $R > 0$. Since $\mathcal{H}_R(\delta_0)$ is compactly imbedded in L^2 (see Theorem 2 or see the details in Andrés and Muñoz^{31, p51}), then there exists a subsequence of $(h_\delta)_\delta$, which will be also denoted by $(h_\delta)_\delta$, that converges to the function h strongly in $L^2(\Omega)$.

Concerning the states the analysis of the previous sections ensures the sequence $(u_\delta)_\delta$ (or a subsequence of it) converges strongly in L^2 to a function $u \in H_0^1(\Omega)$.

Consequently, Theorem 5 states the pair $(h, u) \in \mathcal{A}_{loc}$.

Next, we analyze the optimality of (h, u) . In order to do that we consider $(\tilde{u}_\delta)_\delta$ the sequence of solutions for the nonlocal state Equation 16 when we take h as design. Then $(h, \tilde{u}_\delta) \in \mathcal{A}$. By using Andrés and Muñoz,^{42, Th. 1.4} we can assert that

$$\lim_{\delta \rightarrow 0} B_h(\tilde{u}_\delta - u, \tilde{u}_\delta - u) = 0.$$

Moreover,

$$\lim_{\delta \rightarrow 0} B_h(\tilde{u}_\delta, \tilde{u}_\delta) = \lim_{\delta \rightarrow 0} B_h(u, u) = \int_\Omega |\nabla u(x, t)|^2 dx.$$

By using this identity and the dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{\delta \rightarrow 0} J_\delta(h_\delta, u_\delta) &\leq \lim_{\delta \rightarrow 0} J_\delta(h, \tilde{u}_\delta) \\ &= \lim_{\delta \rightarrow 0} \int_0^T B(\tilde{u}_\delta, \tilde{u}_\delta) dt + cB_0(h, h) \\ &= \int_0^T \int_\Omega |\nabla u(x, t)|^2 dx dt + cB_0(h, h) \\ &= J_{loc}(h, u). \end{aligned}$$

For the reverse estimation, we use Ponce inequality (see Ponce¹⁹ and Andrés and Muñoz^{31, Appendix A, p48}); that is, for any $t \in (0, T)$,

$$\lim_{\delta \rightarrow 0} B(u_\delta, u_\delta) \geq \int_\Omega |\nabla u(x, t)|^2 dx.$$

By means of the above inequality and by Fatou Lemma to arrive at

$$\begin{aligned} \lim_{\delta \rightarrow 0} J_\delta(h_\delta, u_\delta) &= \lim_{\delta \rightarrow 0} \int_0^T B(u_\delta, u_\delta) dt + c \lim_{\delta \rightarrow 0} B_0(h_\delta, h_\delta) \\ &\geq \int_0^T \int_\Omega |\nabla u(x, t)|^2 dx dt + c \lim_{\delta \rightarrow 0} B_0(h, h) \\ &= J_{loc}(h, u). \end{aligned}$$

Thus, we have proved

$$\lim_{\delta \rightarrow 0} J_\delta(h_\delta, u_\delta) = J_{loc}(h, u).$$

Also, it is automatic to check that if $(h^*, u^*) \in \mathcal{A}_{loc}$ and u_δ^* is the underlying state of h^* then, as above, $u_\delta^* \rightarrow u^*$ in L^2 and

$$\begin{aligned} \lim_{\delta \rightarrow 0} J_\delta(h_\delta, u_\delta) &\leq \lim_{\delta \rightarrow 0} J_\delta(h^*, u_\delta^*) \\ &= J_{loc}(h^*, u^*). \end{aligned}$$

Therefore,

$$J_{loc}(h, u) \leq J_{loc}(h^*, u^*)$$

for any $(h^*, u^*) \in \mathcal{A}_{loc}$, namely, (h, u) is optimal for the problem (D_{loc}) . This completes the proof of the

Theorem 6. *Let $(h_\delta)_\delta$ be a sequence of optimal designs for the problems (D_δ) and let $(u_\delta)_\delta$ be their underlying states. Then there exists a pair $(h, u) \in \mathcal{A}_{loc}$ and a subsequence of δ 's such that*

1.

$$\begin{aligned} h_\delta &\rightarrow h \text{ a.e. in } \Omega, \\ u_\delta &\rightarrow u \text{ strongly in } L^2(\Omega). \end{aligned}$$

2.

$$\lim_{\delta \rightarrow 0} J_\delta(h_\delta, u_\delta) = J_{loc}(h, u).$$

3. *The pair (h, u) is a solution of the local design problem (D_{loc}) defined in (31)-(33).*

The same approach serves to approximate several types of problems of optimization. For instance, Theorem 6 applies to problems like

$$J_{loc}(h, u) = \alpha \int_0^T \left(B(u, u) + \|u\|_{L^2(\Omega)}^2 \right) dt + \beta B_0(h, h)$$

or

$$J_{loc}(h, u) = \alpha \int_0^T (B(u, u) + gu) dt + \beta B_0(h, h)$$

with $(h, u) \in \mathcal{A}_{loc}$. Here, α and β are given positive constants and g is any function from $L^2((0, T); L^2(\Omega))$.

We can generalize Theorem 6 to problems under the format

$$J_{loc}(h, u) = \alpha \int_0^T B(u, u) dt + \int_0^T \int_\Omega G(x, u) dx dt + \beta B_0(h, h)$$

where $G : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is any measurable function such that $G(x, \cdot)$ is uniformly Lipschitz continuous.

Concerning the type of problems (D_{loc}^R) , the outcome we obtain is identical to the above case. We have the following:

Theorem 7. *If (h_δ, u_δ) is a sequence of solutions for the problem (D_δ^R) , then we can find $(h, u) \in \mathcal{A}_{loc}^R$ and a subsequence of δ 's such that*

1. $h_\delta \rightarrow h$ a.e. in Ω , $u_\delta \rightarrow u$ strongly in $L^2(\Omega)$
- 2.

$$\lim_{\delta \rightarrow 0} J_\delta^R(h_\delta, u_\delta) = J_{loc}^R(h, u).$$

3. (h, u) is a minimizer of the problem (D_{loc}^R) .

Concerning the generalization of this theorem, the same comments of the problem (D_{loc}) applies in the present setting.

4.4 | Conclusions

At first sight, the above discussion gives an explicit answer about how to solve or approximate some classical optimal design problems. Despite these results concern a matter that, by itself, is of great importance, our main goal is to give some information about the minimizers or minimizing sequences of the problem (D_0) . This goal has only been partially achieved. Two alternatives have been explored:

1. One approach in dealing with (D_0) relies on a penalization of the cost functional. This has been done in (D_δ) . In the limit, (D_δ) has given rise to (D_{loc}) , which is, in a certain sense, a regularization of the original problem (D_0) . Consequently, it seems appropriate to simulate the solutions of (D_δ) when the coefficient of the perturbation c tends to 0. This is a method that could allow us to detect meaningful information about the minimization problem (D_0) .
2. The second way to tackle (D_0) is by means a modification of its set of admissibility. This is achieved by formulating (D_δ^R) and analyzing the limit of its solution. The resultant problem, (D_{loc}^R) , is somehow a regularization of (D_0) . Thus, the analysis of the solutions of (D_δ^R) when $R \rightarrow +\infty$ could provide interesting information about the optimization of (D_0) .

In either case, the numerical simulations performed under these type of hypotheses on the control and the comparison to existing computations from the bibliography could help in going a step further about the idea of regularization we proposed at the beginning of the paper.

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APPENDIX A

Theorem 8 (see previous works^{39-41,49}). Let $V \subset H \subset V'$ be a Gelfand triplet. If $u \in L^2(0, T, V)$ and $u_t \in L^2(0, T, V')$ then $C([0, T]; H)$. Moreover,

1. For any $v \in V$, the real-valued function $t \rightarrow (u, v)_H$ is weakly differentiable in $(0, T)$ and

$$\frac{d}{dt}(u, v)_H = \langle u_t, v \rangle.$$

2. The real-valued function $t \rightarrow \|u(t)\|_{L^2(\Omega_\delta)}^2$ is weakly differentiable in $(0, T)$ and

$$\frac{1}{2} \frac{d}{dt} (\|u(t)\|_H^2) = \langle u_t(t), u(t) \rangle.$$

3. There is a constant $C = C(T)$ such that

$$\|u(t)\|_{L^\infty(0,T;H)} \leq C (\|u(t)\|_{L^2(0,T;V)} + \|u_t\|_{L^2(0,T;V')}).$$