On the Convergence of a Class of Nonlocal Elliptic Equations and Related Optimal Design Problems

Fuensanta Andrés & Julio Muñoz

Journal of Optimization Theory and Applications

ISSN 0022-3239

J Optim Theory Appl
DOI 10.1007/s10957-016-1021-z

Vol. 171, No. 1

**ONLINE
FIRST**

**JOURNAL OF OPTIMIZATION
THEORY AND APPLICATIONS**

Springer

Springer

Your article is protected by copyright and all rights are held exclusively by Springer Science +Business Media New York. This e-offprint is for personal use only and shall not be self-archived in electronic repositories. If you wish to self-archive your article, please use the accepted manuscript version for posting on your own website. You may further deposit the accepted manuscript version in any repository, provided it is only made publicly available 12 months after official publication or later and provided acknowledgement is given to the original source of publication and a link is inserted to the published article on Springer's website. The link must be accompanied by the following text: "The final publication is available at link.springer.com".

On the Convergence of a Class of Nonlocal Elliptic Equations and Related Optimal Design Problems

Fuensanta Andrés¹ · Julio Muñoz¹

Received: 15 March 2016 / Accepted: 1 October 2016
© Springer Science+Business Media New York 2016

Abstract A convergence result for a nonlocal differential equation problem is proved. As a by-product, some results about the convergence for a type of nonlocal optimal design are given. Since these problems give rise to local design problems in the limit, different results on classical existence are obtained as well. Concerning the nonlocal formulation, the state equation is of nonlocal elliptic type and the cost functional we analyze includes, among other cases, an approximation of the square of the gradient.

Keywords Approximation of partial differential equations · Optimal control · Integral equations · G-convergence

Mathematics Subject Classification 35J20 · 49J22 · 45A05 · 46N20

1 Introduction

It seems that a proper treatment of phenomena in mechanics and other interesting fields of applied mathematics can be given through a nonlocal formulation. Indeed, there are a large number of works where this type of modelization has been successfully employed ([1–8]). In particular, it must be underlined that the approximation theory is one side of the nonlocal formulation that has recently attracted an increasing interest. It pursues

Communicated by Giuseppe Buttazzo.

✉ Julio Muñoz
Julio.Munoz@uclm.es

Fuensanta Andrés
Fuensanta.Andres@uclm.es

¹ Universidad de Castilla-La Mancha, Toledo, Spain

the approximation of solutions of local problems by means of nonlocal models. Several works have addressed the study of the asymptotic behavior for nonlocal optimization problems with the aim of understanding the optimality configurations for local classical problems. The reader can look at [9–19], among others. A more specific framework for optimization is nonlocal optimal design and its application to the approximation of local optimal design problems. This is the topic we shall address here.

Accordingly, the main motivation in the present work is the approximation of local classical optimal design problems, for which a result on existence of solutions is quite complicated to derive. The difficulty for these local problems is due to the strict requirements gathered in the formulation of the state equation. A broad perspective of this matter can be found in [20–24]. We propose, as an alternative, the nonlocal framework. This type of model supplies an easier way to guarantee the existence of solutions, and consequently, as we shall see later, it provides a method of approximation of solutions in some specific classical settings. Recent progress has been made in this direction. Some references on this field are [25–29].

In the present contribution we formulate a collection of nonlocal optimal design problems. This collection is parametrized by the called horizon parameter. The state equation of the problem is a nonlocal elliptic one that depends on horizon parameter. For each parameter we prove existence of optimal designs. We also consider its corresponding sequence of nonlocal states. One of our main goals is to prove convergence of the pairs of controls and states to a pair that solves the corresponding local state equation. We are also interested in the consequences that arise from this convergence.

In other words, the goals are the G -convergence analysis of the nonlocal elliptic state equation and its implications in order to face the study of the local optimal design problem.

As we shall see below, the nonlocal problem consists of a nonlocal elliptic state equation that includes a diffusive term and a sequence of kernels converging to a Dirac measure as the horizon parameter tends to zero. These kernels are radial functions fulfilling specific properties, and they serve to model the nonlocal interaction (see Sect. 2.1).

Two types of problems are analyzed. In the first type (Sect. 5) the cost functional is of a nonlocal nature and has two terms (see (10) below): The first term is given by a nonlocal approximation of the norm of the gradient of the state. The second term is also a nonlocal functional acting upon the design. This term plays an important role in the G -convergence of the functional associated with the state equation. We take into account that sequences of designs participating in this process of G -convergence are minimizing sequences. Therefore, in addition to the precompactness of the states derived from the state equation, the second term of the cost provides the same property to be held by the sequence of minimizing designs. In summary, we can say that the structure of the problem we shall face strongly helps in the identification of a limit problem of local nature for which we shall try to prove existence of optimal designs (see (3)–(7) below).

For the second type of problems (Sect. 6), the analysis is identical to the first case. The only difference hinges in the fact that the role of the second term of the cost is now essentially played by a new nonlocal constraint on the controls.

From our analysis, we achieve new existence results for a broad class of local optimal design problems. Furthermore, we must emphasize the fact that the local problems for which we ensure existence of solutions can be viewed as a type of regularization of problems of optimal control lacking classical solution. Moreover, since the way to solve these regularized local problems is by means of the corresponding nonlocal problems, we propose such nonlocal problems, in some sense, as regularizations of the local problem lacking solution.

2 Preliminaries

2.1 Notation and Hypotheses

In the following we make precise the ambient space where we shall work.

1. The domain Ω is a smooth bounded set in $\mathbb{R}^N$ and $\Omega_\delta = \Omega \cup \{\cup_{p \in \partial\Omega} B(p, \delta)\}$, where $B(x, r)$ is the notation for an open ball centered at $x \in \mathbb{R}^N$ and radius $r > 0$, and $\delta \leq \delta_0$ where $\delta_0 > 0$ is a given small number.
2. About the source term we assume

$$f \in L_0^2(\Omega_\delta) \doteq \left\{ u : \Omega_\delta \rightarrow \mathbb{R} \mid u \in L^2(\Omega), u = 0 \text{ in } \Omega_\delta - \Omega \right\}.$$

3. The set of designs is

$$\mathcal{H} \doteq \left\{ h : \Omega_\delta \rightarrow \mathbb{R} \mid h(x) \in [h_{\min}, h_{\max}] \text{ a.e. } x \in \Omega, \right. \\ \left. h = 0 \text{ in } \Omega_\delta - \Omega, \int_{\Omega} h(x) \, dx = V_0 \right\}$$

where $h_{\min}$, $h_{\max}$ and V_0 are positive constants such that $0 < h_{\min} |\Omega| < V_0 < h_{\max} |\Omega|$.

4. The kernels $(k_\delta)_{\delta \leq \delta_0}$ is a family of radial functions such that for every $\delta \leq \delta_0$ hold the following properties:
 - (a)

$$\frac{1}{N} \int_{B(0, \delta)} k_\delta(|s|) \, ds = 1$$

- (b)

$$\text{supp } k_\delta \subset B(0, \delta)$$

- (c) $k_\delta \geq 0$ and the following additional properties hold:

- i. the function $K_r : B(0, \delta) - \{0\} \rightarrow [0, +\infty[$ defined by the formula $K_r(z) \doteq \frac{k_\delta(|z|)}{|z|}$ is integrable, i.e., $K_r \in L^1(B(0, \delta))$.

- ii. the function $K_s : B(0, \delta) - \{0\} \rightarrow [0, +\infty[$ defined by the formula $K_s(z) \doteq \frac{k_\delta(|z|)}{|z|^2}$ is assumed to be singular near the origin in the sense that

$$\lim_{\theta \rightarrow 0^+} \int_{B(0, \delta) - B(0, \theta)} K_s(z) dz = +\infty.$$

We also need to specify the functional space X_0 we shall deal with. We define the space $X = X(\delta)$ as

$$X = \left\{ u \in L_0^2(\Omega_\delta) : B(u, u) < \infty \right\}, \quad (1)$$

where

$$B(u, v) = \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{k_\delta(|x' - x|)}{|x' - x|^2} (u(x') - u(x)) (v(x') - v(x)) dx' dx. \quad (2)$$

We define the space X_0 as

$$X_0 = \text{cl} \left(C_{co}^\infty(\Omega_\delta) \right)$$

where

$$C_{co}^\infty(\Omega_\delta) = \left\{ f : \Omega_\delta \rightarrow \mathbb{R} : f \in C_c^\infty(\Omega) \text{ and } f = 0 \text{ in } \Omega_\delta - \Omega \right\} \subset X$$

and $\text{cl} \left(C_{co}^\infty(\Omega_\delta) \right)$ is the closure with respect to the norm $\|\cdot\|$ given in X via the quadratic form $B(\cdot, \cdot)$, that is

$$X_0 = \left\{ v \in X : \text{there is } (v_j) \subset C_{co}^\infty(\Omega_\delta) \text{ such that } \lim_j B(v_j - v, v_j - v) = 0 \right\}.$$

It must be recalled that for each $h \in \mathcal{H}$ and each $\delta \geq \delta_0$ fixed, X_0 is a Hilbert space with the inner product $B_h(\cdot, \cdot)$. In this context it will be useful to take into account the existence of a certain basis of eigenfunctions (see “Appendices A and B”).

Next, we describe the problems, both local and nonlocal, and we give a detailed account of the results contained in this paper.

2.2 The Local Design Problem

The final goal is to solve or approximate the minimization problem $(\mathcal{P}_{loc})$ described as follows:

$$\inf_{(h, u) \in \mathcal{A}_{loc}} J_{loc}(h, u) \quad (3)$$

where, the cost functional is

$$J_{loc}(h, u) \doteq \int_{\Omega} |\nabla u|^2 dx + c B_0(h, h) \quad (4)$$

c is a fixed positive constant,

$$B_0(h, h) = \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} k_{\delta_0}(|x' - x|) \frac{|h(x') - h(x)|^2}{|x' - x|^2} dx' dx, \quad (5)$$

k_{δ_0} is a fixed kernel which is assumed to verify all the conditions given in the above hypothesis (4) and $\delta_0 > 0$ is a fixed, small given number. The admissibility set is

$$\mathcal{A}_{loc} = \left\{ (h, u) \in \mathcal{H} \times H_0^1(\Omega) : (h, u) \text{ solves (7)} \right\} \quad (6)$$

and the local state equation is

$$-\operatorname{div}(h \nabla u) = f, u \in H_0^1(\Omega). \quad (7)$$

In connection with this equation we shall consider the bilinear form $b_h(\cdot, \cdot)$ defined in $H^1(\Omega) \times H^1(\Omega)$ through the formula

$$b_h(u, v) = \int_{\Omega} h(x) \nabla u(x) \nabla v(x) dx. \quad (8)$$

Eventually, in Sect. 6 the set $\mathcal{H}$ will be modified by adding a nonlocal constraint and the constant c will be set to zero.

2.3 The Nonlocal Design Problem

Even though our goal is to be able to infer some information about the above design problem, the starting point in our investigation will be the following minimization problem $(\mathcal{P}_{\delta})$:

$$\inf_{(h, u) \in \mathcal{A}_{\delta}} J_{\delta}(h, u) \quad (9)$$

where the cost functional is defined as

$$J_{\delta}(h, u) \doteq \int_{\Omega_{\delta}} \int_{\Omega_{\delta}} k_{\delta}(|x' - x|) \frac{(u(x') - u(x))^2}{|x' - x|^2} dx' dx + c B_0(h, h), \quad (10)$$

and c is a fixed positive constant. The admissibility set is

$$\mathcal{A}_{\delta} = \{(h, u) \in \mathcal{H} \times X_0 : (h, u) \text{ solves (12)}\} \quad (11)$$

and the nonlocal state equation is

$$B_h(u, w) = (f, w)_{L^2(\Omega) \times L^2(\Omega)} \quad (12)$$

for any w in X_0 , that is

$$\int_{\Omega_\delta} \int_{\Omega_\delta} H(x', x) k_\delta(|x' - x|) \frac{(u(x') - u(x))(w(x') - w(x))}{|x' - x|^2} dx' dx = \int_{\Omega_\delta} f w dx \quad (13)$$

for any $w \in X_0$, where $H(x', x) \doteq \frac{h(x') + h(x)}{2}$.

As in the local setting, in Sect. 6 we shall address the case $c = 0$ with $\mathcal{H}$ including a nonlocal constraint.

2.4 Results and Organization of the Paper

As we have already mentioned, our purpose is to state the existence of solutions for the family of problems $(\mathcal{P}_\delta)$, defined via (9)–(13), and then, when $\delta \rightarrow 0$, to see whether this sequence of solutions converges to the solution of the problem $(\mathcal{P}_{loc})$ given by (3)–(7). These tasks obviously entail several issues or steps.

1. We need existence and uniqueness for the state Eq. (12). This matter has already been completely treated and solved. [12, 17, 30, 31] and [13] are some appropriate references for our framework.

For the sake of clarity, we shall recall a couple of results that will be useful in our research. The first one has to do with the existence of solutions. It is Theorem 1.2 from [17] which establishes (for each $\delta > 0$ and h fixed) the existence and uniqueness for the nonlocal Eq. (12).

The second result is related to the convergence of solutions of $(\mathcal{P}_\delta)$ when $\delta \rightarrow 0$ and h is fixed. Theorem 1.4 from [17] states the following: If $(u_\delta)_\delta$ is the sequence of solutions of (12), then there exists $u \in H_0^1(\Omega)$ and a subsequence of $(u_\delta)_\delta$ (which will be also denoted by $(u_\delta)_\delta$) that converges strongly in $L^2(\Omega)$ to u . In addition this function u , jointly with h , is the solution of the local problem (7).

2. We must analyze the G -convergence of (12). We study the convergence of the solutions of the Eqs. (12) toward the solution of the local state equation defined in (7). This issue is analyzed in Sect. 3, and the main result is Theorem 3.1.

In order to overcome the difficulties related to the pass to the limit in the nonlocal equation, we have assumed the pointwise convergence of the designs $(h_\delta)_\delta$ to a certain function h . This fact together with certain compactness results (see [17, 32, 33],) makes possible the G -convergence to the elliptic functional that defines Eq. (7).

3. We have to guarantee there exists at least one solution of the nonlocal optimal design problem (9)–(12). This point is an intermediate step in the present manuscript that has been already treated in [29]. Related to this issue we state Theorem 4.1 in Sect. 4.
4. The next step is to see how the minimizers of $(\mathcal{P}_\delta)$ converge to the minimizer of $(\mathcal{P}_{loc})$: See Theorem 5.1 in Sect. 5. This result is accompanied by a generalized theorem where we consider a broad class of cost functionals (see Theorem 5.2).
5. Section 6 is devoted to the study of the situation when $c = 0$. This time a nonlocal bound on the controls is imposed. Namely, we shall consider an admissibility set

that includes the constraint

$$\int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} k_{\delta_0} (|x' - x|) \frac{|h(x') - h(x)|^2}{|x' - x|^2} dx' dx < \varepsilon, \quad (14)$$

where ε is a fixed, positive number. The results of Sect. 6 are summarized in Theorems 6.1 and 6.2.

6. Section 7 contains some perspectives derived from the results of the paper. In this section, we comment how the resulting nonlocal optimal design problems ((9)–(13) or (32)–(35)) are in some sense, two regularizations of the celebrated problem defined by

$$\min_{h \in \mathcal{H}} \int_{\Omega} |\nabla u|^2 dx, \quad (15)$$

subject to

$$-\operatorname{div} (h \nabla u) = f \text{ in } H_0^1(\Omega). \quad (16)$$

7. A couple of appendices are devoted to giving two key technical results.

3 G-convergence

Let $(h_\delta)_\delta$ be a sequence of functions from the set

$$L_0^\infty(\Omega_\delta) = \{v : \Omega_\delta \rightarrow \mathbb{R} : v \in L^\infty(\Omega), v = 0 \text{ in } \Omega_\delta - \Omega\}$$

such that

$$h_\delta(x) \rightharpoonup h(x) \text{ weak-}^* \text{ in } L^\infty(\Omega) \text{ if } \delta \rightarrow 0. \quad (17)$$

For the weak- $*$ limit $h \in L_0^\infty(\Omega_\delta)$, we consider its corresponding local state $u = u(h)$, that is, the only solution of the equation

$$-\operatorname{div}(h \nabla w) = f, \quad w \in H_0^1(\Omega).$$

Proposition 3.1 *If we assume hypothesis (17), then*

$$\liminf_{\delta \rightarrow 0} \min_{v \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(v, v) - (f, v) \right\} \leq \frac{1}{2} b_h(u, u) - (f, u) \quad (18)$$

or equivalently

$$\liminf_{\delta \rightarrow 0} \min_{v \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(v, v) - (f, v) \right\} \leq \min_{w \in H_0^1(\Omega)} \left\{ \frac{1}{2} b_h(w, w) - (f, w) \right\}. \quad (19)$$

Proof Let u_δ be the sequence defined by

$$u_\delta = \arg \min \left\{ \frac{1}{2} B_{h_\delta} (v, v) - (f, v) : v \in X_0 \right\}.$$

We clearly have

$$b_h (u, u) = 2 B_{h_\delta} (u_\delta, u) - b_h (u, u)$$

and thereby

$$\begin{aligned} B_{h_\delta} (u_\delta, u_\delta) - b_h (u, u) &= B_{h_\delta} (u_\delta, u_\delta) - 2 B_{h_\delta} (u_\delta, u) + (b_h (u, u) - B_h (u, u)) \\ &\quad + B_{h_\delta} (u, u) + (B_h (u, u) - B_{h_\delta} (u, u)). \end{aligned} \quad (20)$$

The one hand we know

$$\lim_{\delta \rightarrow 0} (b_h (u, u) - B_h (u, u)) = 0. \quad (21)$$

(see proof of Theorem 8 in [29]). On the other, by using

$$h_\delta - h \rightarrow 0 \text{ weak } * \text{ in } L^\infty$$

and

$$\int_\Omega \frac{k_\delta (|x' - x|)}{|x' - x|^2} (u' - u)^2 \, dx' \rightarrow |\nabla u (x)|^2 \text{ strongly in } L^1 (\Omega)$$

(see [34, Corollary 1])¹, it is straightforward to realize that

$$\lim_{\delta \rightarrow 0} (B_h (u, u) - B_{h_\delta} (u, u)) = \lim_{\delta \rightarrow 0} (B_{h-h_\delta} (u, u)) = 0.$$

By using (20), (21) and the above limit, we finally obtain

$$\lim_{\delta \rightarrow 0} (B_{h_\delta} (u_\delta, u_\delta) - b_h (u, u)) = \lim_{\delta \rightarrow 0} B_{h_\delta} (u_\delta - u, u_\delta - u) \geq 0,$$

¹ Here we are using $\frac{1}{|S^{N-1}|} \int_{S^{N-1}} (\omega \cdot \mathbf{e})^2 \, d\sigma (\omega) = \frac{1}{N}$, where σ stands for the $(N-1)$ -dimensional Hausdorff measure on the unit sphere S^{N-1} and $\mathbf{e}$ is any unit vector in $\mathbb{R}^N$.

and consequently (18) holds. In addition, we confirm (19) because

$$\begin{aligned} \liminf_{\delta \rightarrow 0} \min_{v \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(v, v) - (f, v) \right\} &= \liminf_{\delta \rightarrow 0} \left(\frac{1}{2} B_{h_\delta}(u_\delta, u_\delta) - (f, u_\delta) \right) \\ &= \liminf_{\delta \rightarrow 0} \left(-\frac{1}{2} B_{h_\delta}(u_\delta, u_\delta) \right) \\ &\leq -\frac{1}{2} b_h(u, u) \\ &= \frac{1}{2} b_h(u, u) - (f, u) \\ &= \min_{w \in H_0^1(\Omega)} \left\{ \frac{1}{2} b_h(w, w) - (f, w) \right\}. \end{aligned}$$

□

Proposition 3.2 Assume the pointwise convergence

$$\lim_{\delta \rightarrow 0} h_\delta(x) = h(x) \text{ a.e. } x \in \Omega. \quad (22)$$

If $u^* \in H_0^1$ is the strong limit in $L^2(\Omega)$ of the sequence $(u_\delta)_\delta$, then

$$\lim_{\delta} B_{h_\delta}(u_\delta, u_\delta) \geq b_h(u^*, u^*) \quad (23)$$

and

$$\lim_{\delta} \min_{v \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(v, v) - (f, v) \right\} \geq \min_{w \in H_0^1(\Omega)} \left\{ \frac{1}{2} b_h(w, w) - (f, w) \right\}. \quad (24)$$

Proof We firstly recall the following fact: Since $(u_\delta)_\delta$ is the sequence of solutions of

$$\min_{w \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(w, w) - (f, w) \right\}$$

then $B_{h_\delta}(u_\delta, u_\delta)$ is uniformly bounded. Under these circumstances we know there exists a function $u^* \in H_0^1(\Omega)$ such that $(u_\delta)_\delta$ (or at least a subsequence) converges strongly to u^* in $L^2(\Omega)$ (see [32]).

We examine the quadratic functional

$$B_{h_\delta}(u_\delta, u_\delta) \doteq \int_{\Omega_\delta} \int_{\Omega_\delta} H_\delta k_\delta(|x' - x|) \frac{(u_\delta(x') - u_\delta(x))^2}{|x' - x|^2} dx' dx$$

Let us consider now the sequence of measures defined for every measurable set $A \subset \Omega$ by:

$$\mu_\delta(A) = \int_A \left(\int_{B(x, \delta)} k_\delta(|x' - x|) \frac{(u_\delta(x') - u_\delta(x))^2}{|x' - x|^2} dx' \right) dx.$$

We note

$$B_{h_\delta}(u_\delta, u_\delta) = \int_{\Omega} h_\delta(x) \, d\mu_\delta(x)$$

and for any measurable set A , we have

$$\liminf_{\delta \rightarrow 0} \mu_\delta(A) \geq \liminf_{\delta \rightarrow 0} \int_A \int_A k_\delta(|x' - x|) \frac{(u_\delta(x') - u_\delta(x))^2}{|x' - x|^2} \, dx' \, dx.$$

By using now Ponce's inequality for measurable sets (see "Appendix A"), we get the following setwise inequality

$$\liminf_{\delta \rightarrow 0} \mu_\delta(A) \geq \int_A |\nabla u^*(x)|^2 \, dx.$$

If we define the measure μ_0 as

$$\mu_0(A) = \int_A |\nabla u^*(x)|^2 \, dx,$$

then the previous discussion yields the estimate

$$\liminf_{\delta \rightarrow 0} \mu_\delta(A) \geq \mu_0(A).$$

By taking into account the above inequality and the pointwise convergence $h_\delta \rightarrow h$ a.e. in Ω , and by using the Generalized Fatou's Lemma ([36, 37]), we arrive at

$$\begin{aligned} \liminf_{\delta \rightarrow 0} B_{h_\delta}(u_\delta, u_\delta) &= \liminf_{\delta \rightarrow 0} \int_{\Omega} h_\delta(x) \, d\mu_\delta(x) \\ &\geq \int_{\Omega} h(x) \, d\mu_0(x) \\ &= \int_{\Omega} h(x) |\nabla u^*(x)|^2 \, dx, \end{aligned}$$

which is the thesis (23). Finally, thanks to the inequality (23), it is immediate to deduce (24):

$$\begin{aligned} \lim_{\delta \rightarrow 0} \min_{v \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(v, v) - (f, v) \right\} &= \lim_{\delta \rightarrow 0} \frac{1}{2} B_{h_\delta}(u_\delta, u_\delta) - \lim_{\delta \rightarrow 0} (f, u_\delta) \\ &\geq \frac{1}{2} \int_{\Omega} h(x) |\nabla u^*(x)|^2 \, dx - (f, u^*) \\ &\geq \min_{w \in H_0^1(\Omega)} \left\{ \frac{1}{2} b_h(w, w) - (f, w) \right\}. \end{aligned}$$

□

As a corollary of the inequalities (19) and (24), we immediately arrive at the main result of this section:

Theorem 3.1 (G-Convergence) *Assume $(h_\delta)_\delta \subset L_0^\infty(\Omega_\delta)$, $h \in L_0^\infty(\Omega_\delta)$, such that (17) and (22) are fulfilled. If $u^* \in H_0^1$ is the strong limit in L^2 of the sequence $(u_\delta)_\delta$, and u is the underlying state of h , then*

$$\lim_{\delta \rightarrow 0} \min_{v \in X_0} \left\{ \frac{1}{2} B_{h_\delta}(v, v) - (f, v) \right\} = \min_{w \in H_0^1(\Omega)} \left\{ \frac{1}{2} b_h(w, w) - (f, w) \right\}$$

and $u = u^*$, that is

$$\min_{w \in H_0^1(\Omega)} \left\{ \frac{1}{2} b_h(w, w) - (f, w) \right\} = \frac{1}{2} \int_{\Omega} h(x) |\nabla u^*(x)|^2 dx - (f, u^*).$$

4 Existence for the Nonlocal Design Problem

We deal with problem $(\mathcal{P}_\delta)$ determined by (9)–(13). We basically follow the same lines as in [29].

Assume δ is fixed. Let $(h_j)_j$ be a minimizing sequence of controls, and let $(u_j)_j$ be the associated states defined as the solution of the Eq. (13). We know the sequence $(h_j)_j$ is uniformly bounded in L^∞ so that there is a subsequence of j 's and a function $h \in L^\infty$ such that $h_j \rightharpoonup h$ weak-* in L^∞ . In particular we are in a position to apply Theorem 6 from [29] to ensure there is a subsequence of $(u_j)_j$, which will be not relabeled, that converges strongly in L^2 to some $u \in X_0$, and such that the pair $(h, u) \in \mathcal{A}_\delta$. We remark that thanks to the convexity of the integrand of the cost we have the lower semicontinuity with respect to the above convergence ([35]). This statement read as

$$\begin{aligned} & \lim_{j \rightarrow \infty} \left(\int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x' - x|) \frac{(u_j(x') - u_j(x))^2}{|x' - x|^2} dx' dx + c B_0(h_j, h_j) \right) \\ & \geq \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x' - x|) \frac{(u(x') - u(x))^2}{|x' - x|^2} dx' dx + B_0(h, h). \end{aligned}$$

and hence (h, u) is a minimizer of (9)–(13).

Theorem 4.1 (Existence of nonlocal optimal designs) *For each $\delta > 0$, there exists a solution h_δ of the minimization problem (9)–(13).*

Remark 4.1 It is automatic to assert that the thesis of the above theorem remains valid if we consider any cost functional with the following format

$$J(h, u) = \alpha B(u, u) + \int_{\Omega_\delta} \int_{\Omega_\delta} F(x', x, u(x'), u(x)) dx' dx + \beta B_0(h, h) \quad (25)$$

where α and β are positive constants and the integrand function $F : \Omega_\delta \times \Omega_\delta \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to satisfy the following conditions

1. $F(\cdot, \cdot, u_1, u_2)$ is measurable for any $(u_1, u_2) \in \mathbb{R} \times \mathbb{R}$;
2. $F(x_1, x_2, \cdot, \cdot)$ is lower semicontinuous function for any $(x_1, x_2) \in \Omega_\delta \times \Omega_\delta$.

5 Convergence to the Local Problem

We analyze the limit of the solution of the problem $(\mathcal{P}_\delta)$ when $\delta \rightarrow 0$. Let $(h_\delta)_\delta$ be a sequence of optimal designs for $(\mathcal{P}_\delta)$, and let $(u_\delta)_\delta$ be the associated states defined as the solution of the Eq. (13). We know the sequence $(h_\delta)_\delta$ is uniformly bounded in L^∞ , and consequently the term $B_0(h_\delta, h_\delta)$ is uniformly bounded too. To prove that, we firstly assert

$$J(h_\delta, u_\delta) \leq J(h^*, u_\delta^*)$$

where h^* is a given regular control function admissible for every problem $(\mathcal{P}_\delta)$, and $(u_\delta^*)_\delta$ is the corresponding sequence of associated states. Since there is $u^* \in H_0^1(\Omega)$ such that $u_\delta^* \rightarrow u^*$ strongly in L^2 and

$$\lim_{\delta \rightarrow 0} B_{h^*}(u_\delta^*, u_\delta^*) = b_{h^*}(u^*, u^*),$$

then $B_{h^*}(u_\delta^*, u_\delta^*)$ is uniformly bounded and, consequently, $B(u_\delta^*, u_\delta^*)$ is uniformly bounded too. By noting that

$$J(h^*, u_\delta^*) = B(u_\delta^*, u_\delta^*) + cB_0(h^*, h^*),$$

and by taking into account that the control h^* is enough regular so that the term $B_0(h^*, h^*)$ is finite, then $J(h^*, u_\delta^*)$ is also uniformly bounded. Consequently, there is a positive constant C such that $J(h_\delta, u_\delta) < C$ uniformly in δ . In particular this bound implies the sequence $B_0(h_\delta, h_\delta)$ is uniformly bounded.

Next, we know the set

$$\mathcal{H}_C(\delta_0) \doteq \{h \in \mathcal{H} : B_0(h, h) < C\}$$

is compactly imbedded in L^2 (here $C > 0$ is a given constant) (see Appendix B). Thus, we ensure there exists $h \in L^2$ and a subsequence of $(h_\delta)_\delta$ such that $h_\delta \rightarrow h$ strongly in $L^2(\Omega)$. In particular, there is a sequence (or at least a subsequence) of $(h_\delta)_\delta$ that converges pointwise to h in Ω .

Concerning the states, we also have a uniform bound for $B_{h_\delta}(u_\delta, u_\delta)$ and therefore we ensure $(u_\delta)_\delta$ (or a subsequence of it) converges strongly in L^2 to some $u \in H_0^1(\Omega)$ ([17, Theorem 1.4]). By applying Theorem 3.1 we are in a position to state that (h, u) is an admissible pair for the local problem (3)–(7).

Let $(\tilde{u}_\delta)$ be the sequence of states that correspond to h for each problem $(\mathcal{P}_\delta)$. By using the arguments of the proof of Theorem 1.4 from [17], we are going to derive the

identity

$$\lim_{\delta \rightarrow 0} \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta (|x' - x|) \frac{(\tilde{u}_\delta (x') - \tilde{u}_\delta (x))^2}{|x' - x|^2} dx' dx = \int_{\Omega} |\nabla u|^2 dx.$$

Indeed, the above equality is true because Theorem 1.4 from [17] states the strong convergence $\tilde{u}_\delta \rightarrow u$ in X_0 , that is

$$\lim_{\delta \rightarrow 0} B_h (\tilde{u}_\delta - u, \tilde{u}_\delta - u) = 0.$$

Then we have

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta (|x' - x|) \frac{(\tilde{u}_\delta (x') - \tilde{u}_\delta (x))^2}{|x' - x|^2} dx' dx \\ &= \lim_{\delta \rightarrow 0} \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta (|x' - x|) \frac{(u (x') - u (x))^2}{|x' - x|^2} dx' dx, \end{aligned}$$

and the right part of the above identity is precisely $\int_{\Omega} |\nabla u|^2 dx$. Thus, we have obtained

$$\lim_{\delta \rightarrow 0} J_\delta (h_\delta, u_\delta) \leq \lim_{\delta \rightarrow 0} J_\delta (h, \tilde{u}_\delta) = J_{loc} (h, u). \quad (26)$$

We also have the reverse estimate. We use the inequality established by Ponce (see [32, p. 12] and [9, Remark 7 and Lemma 4]), namely

$$\lim_{\delta \rightarrow 0} \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta (|x' - x|) \frac{(u_\delta (x') - u_\delta (x))^2}{|x' - x|^2} dx' dx \geq \int_{\Omega} |\nabla u|^2 dx, \quad (27)$$

and Fatou's Lemma to deduce

$$\begin{aligned} & \lim_{\delta \rightarrow 0} J_\delta (h_\delta, u_\delta) \\ &= \lim_{\delta \rightarrow 0} \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta (|x' - x|) \frac{(u_\delta (x') - u_\delta (x))^2}{|x' - x|^2} dx' dx + c \lim_{\delta \rightarrow 0} B_0 (h_\delta, h_\delta) \\ &\geq \int_{\Omega} |\nabla u|^2 dx + c B_0 (h, h) \\ &= J_{loc} (h, u). \end{aligned} \quad (28)$$

Therefore

$$\lim_{\delta \rightarrow 0} J_\delta (h_\delta, u_\delta) = J_{loc} (h, u).$$

In addition, we have the same inequality as in (26) whichever local control h^* we choose: Indeed, if u^* is its local state, then $u_\delta^* \rightarrow u^*$ strongly in L^2 and

$$\lim_{\delta \rightarrow 0} J_\delta(h_\delta, u_\delta) \leq \lim_{\delta \rightarrow 0} J_\delta(h^*, u_\delta^*) = J_{loc}(h^*, u^*).$$

We arrive then at the optimal inequality $J_{loc}(h, u) \leq J_{loc}(h^*, u^*)$, and we have completed the proof of

Theorem 5.1 (Convergence to the local problem) *Let $(h_\delta)_\delta$ be a sequence of nonlocal optimal designs for the problems $(\mathcal{P}_\delta)$, and let $(u_\delta)_\delta$ be the associated states defined as the solution of the Eq. (13). Then there exists $(h, u) \in \mathcal{A}_{loc}$ and a subsequence of δ 's such that*

1.

$$\begin{aligned} h_\delta &\rightarrow h, \text{ a.e. in } \Omega, \\ u_\delta &\rightarrow u \text{ strongly in } L^2(\Omega). \end{aligned}$$

2.

$$\lim_{\delta \rightarrow 0} J_\delta(h_\delta, u_\delta) = J_{loc}(h, u)$$

3. The pair $(h, u) \in \mathcal{A}_{loc}$ is a solution of the local design problem (3)–(4).

The above approximation result remains valid if we consider a certain type of cost functionals from the class defined in (25). For instance, some particularly interesting problems could be

$$J_\delta(h, u) = \alpha B(u, u) + \|u\|_{L^2}^2 + \beta B_0(h - h_0, h - h_0) \quad (29)$$

or

$$J_\delta(h, u) = \alpha B(u, u) + \int_{\Omega} f(x) u(x) dx + \beta B_0(h - h_0, h - h_0), \quad (30)$$

where h_0 is a given admissible design, and α, β are positive constants.

Theorem 5.2 *Let $G : \Omega_\delta \times \mathbb{R} \rightarrow \mathbb{R}$ be any given uniformly Lipschitz continuous function². There is at least one solution for the design problem defined as*

$$\min_{(h, u) \in \mathcal{A}_{loc}} \alpha \int_{\Omega} |\nabla u(x)|^2 dx + \int_{\Omega_\delta} G(x, u(x)) dx + \beta B_0(h, h). \quad (31)$$

For the proof we just note that the second term can be rewritten as

$$\int_{\Omega_\delta} G(x, u(x)) dx = \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{G(x, u(x')) + G(x, u(x))}{2|\Omega_\delta|} dx' dx.$$

² That is, for any $x \in \Omega_\delta$ and any $u, v \in \mathbb{R}$ there is a constant $L > 0$ such that $|G(x, u) - G(x, v)| \leq L|u - v|$.

Also, it is appropriate to remark that the strong convergence in L^2 of $(u_\delta)_\delta$ makes automatic the pass to the limit both in (26) and (28). Details are straightforward.

6 Existence Under a Nonlocal Constraint on the Control

We explore problem $(\mathcal{P}_\delta)$ given by

$$\inf_{(h,u) \in \mathcal{A}_\delta} J_\delta(h, u) \quad (32)$$

where

$$J_\delta(h, u) = \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x' - x|) \frac{(u(x') - u(x))^2}{|x' - x|^2} dx' dx \quad (33)$$

and

$$\mathcal{A}_\delta = \{(h, u) \in \mathcal{H}_\varepsilon(\delta_0) \times X_0 : (h, u) \text{ solves (12)}\} \quad (34)$$

for

$$\mathcal{H}_\varepsilon(\delta_0) = \{h \in \mathcal{H} : B_0(h, h) < \varepsilon\}, \quad (35)$$

the bilinear operator B_0 defined as in (5), and both ε and δ_0 are given positive constants. We proceed as in the previous section. If $((h_\delta, u_\delta))_\delta$ is the sequence of minimizer of the problem $(\mathcal{P}_\delta)$ given in (32)–(35), then there is a subsequence of $(u_\delta)_\delta$ and a function $u \in H_0^1(\Omega)$ such that $u_\delta \rightarrow u$ strongly in $L^2(\Omega)$. Also, since $(h_\delta)_\delta \subset \mathcal{H}_\varepsilon(\delta_0)$ the term $B_0(h_\delta, h_\delta)$ is uniformly bounded and there exists $h \in L^2$ and a subsequence of $(h_\delta)_\delta$ such that $(h_\delta)_\delta \rightarrow h$ strongly in $L^2(\Omega)$. Therefore we can apply Theorem 3.1 to ensure $(h, u) \in \mathcal{A}_{loc}$, where

$$\mathcal{A}_{loc} = \{(g, w) \in \mathcal{H}_\varepsilon(\delta_0) \times H_0^1(\Omega) : (g, w) \text{ solves (7)}\}. \quad (36)$$

Moreover, the same arguments of the preceding section allow us to state (h, u) is a solution of the optimal design problem $(\mathcal{P}_{loc})$ defined as

$$\inf_{(g,w) \in \mathcal{A}_{loc}} \|\nabla w\|_{L^2(\Omega)}^2 \quad (37)$$

More precisely, we have the following theorem:

Theorem 6.1 *If (h_δ, u_δ) is a sequence of solution for the problem (32)–(35) then there exists $(h, u) \in \mathcal{A}_{loc}$, and a subsequence of δ 's such that*

1. $h_\delta \rightarrow h$, a.e. in Ω and $u_\delta \rightarrow u$ strongly in $L^2(\Omega)$, and

$$\lim_{\delta \rightarrow 0} J_\delta(h_\delta, u_\delta) = \|\nabla u\|_{L^2(\Omega)}^2.$$

2. The pair (h, u) is a solution of the local design problem (37)–(36).

We can now prove this general existence result:

Theorem 6.2 *With G as in Theorem 5.2, there exists a solution of the problem*

$$\min_{(h,u) \in \mathcal{A}_{loc}} \alpha \int_{\Omega} |\nabla u(x)|^2 dx + \int_{\Omega} G(x, u(x)) dx. \quad (38)$$

Once again, the solution achieved in (38) can be determined in the limit by the solutions of the nonlocal design problem $(\mathcal{P}_{\delta})$ defined as

$$\min_{(h,u) \in \mathcal{A}_{\delta}} \alpha B(u, u) + \int_{\Omega_{\delta}} \int_{\Omega_{\delta}} \frac{G(x, u(x')) + G(x, u(x))}{2 |\Omega_{\delta}|} dx' dx.$$

7 Conclusions

Once we have demonstrated the existence for the problem $(\mathcal{P}_{\delta})$ and the convergence to the problem $(\mathcal{P}_{loc})$, we return to the local problem (15)–(16). Some suitable comments concerning this problem should be taken into consideration. In view of the above results we suggest, two different regularizations:

1. Regularization by a penalization of the nonlocal term. After Sect. 5 we can ensure the convergence of problems $(\mathcal{P}_{\delta})$, defined via (9)–(13), to problem $(\mathcal{P}_{loc})$ given by (3)–(7). Accordingly, the type of regularization we suggest for problem (15)–(16) is $(\mathcal{P}_{\delta})$ with δ and c small enough.
2. Regularization by adding a nonlocal constraint in the set of admissibility. Thanks to the results of Sect. 6, problem $(\mathcal{P}_{\delta})$, defined by (32)–(35), approximates (37)–(36) if δ is small. Consequently, $(\mathcal{P}_{\delta})$, with δ small and ε big enough, can be considered a nonlocal regularization of (15)–(16).

Consequently, as a part of future works, the study we propose is to compare the nonlocal solutions of the above two regularizations to the approximation results of optimality we know for the local problem (15)–(16). This comparison is nevertheless a goal that cannot be accomplished without difficulties. Although there is an abundant numerical literature for this local problem (see for instance [20, 22, 38, 39]), we must point out that the numerical efficiency for nonlocal models is still in a initial step of research. This fact clearly entails a new challenge that we should face.

Acknowledgments We are profoundly grateful to P. Pedregal for his encouragement and helpful comments. We also want to thank J. C. Bellido for interesting discussions on the subject.

Appendix A: A Nonlocal Inequality for Measurable Sets

We analyze the limit

$$\lim_{\delta \rightarrow 0} \int_G \int_G k_{\delta}(|x' - x|) \frac{(u_{\delta}(x') - u_{\delta}(x))^2}{|x' - x|^2} dx' dx$$

where $G \subset \Omega$. We assume $u_\delta \rightarrow u^*$ strongly in $L^2(\Omega)$ where $(u_\delta) \subset X_0$, $u^* \in H_0^1(\Omega)$, and we assume the sequence

$$\xi_\delta(x', x) \doteq k_\delta(|x' - x|) \frac{(u_\delta(x') - u_\delta(x))^2}{|x' - x|^2}$$

is uniformly bounded in $L^1(\Omega \times \Omega)$.

It is well known that, at least for a subsequence of (u_δ) , the inequality

$$\lim_{\delta \rightarrow 0} \int_G \int_G k_\delta(|x' - x|) \frac{(u_\delta(x') - u_\delta(x))^2}{|x' - x|^2} dx' dx \geq \int_G |\nabla u^*(x)|^2 dx \quad (39)$$

holds for every open and bounded $G \subset \Omega$ (see [32, p. 12]).

We prove the above inequality for any measurable set $G \subset \Omega$. Since ξ_δ is uniformly bounded in $L^1(\Omega \times \Omega)$, we can apply Chacon's biting Lemma (see [40]): there exists a subsequence, not relabeled, a nonincreasing sequence of measurable sets $G_n \subset \Omega \times \Omega$, $|G_n| \searrow 0$ and $\xi \in L^1(\Omega \times \Omega)$ such that

$$\xi_\delta \rightharpoonup \xi \text{ in } L^1(\Omega \times \Omega \setminus G_n) \quad (40)$$

for all n .

We take into account the identity

$$\int_G |\nabla u^*(x)|^2 dx = \iint_{G \times G} \frac{|\nabla u^*(x')|^2 + |\nabla u^*(x)|^2}{2|G|} dx' dx,$$

and we shall prove for every measurable A

$$\iint_{A \times A} \xi(x', x) dx' dx \geq \iint_{A \times A} U^*(x', x) dx' dx = \int_A |\nabla u^*(x)|^2 dx \quad (41)$$

where $U^*(x', x) \doteq \frac{|\nabla u^*(x')|^2 + |\nabla u^*(x)|^2}{2|A|}$. We prove (41) by a contradiction argument. We assume there is a measurable set A such that

$$\iint_{A \times A} \xi dx' dx < \iint_{A \times A} U^* dx' dx.$$

Thereby, we can assume

$$\iint_{A \times A} \xi dx' dx < \iint_{A \times A} U^* dx' dx$$

where $A \subset \Omega$ is an open set such that $|A| > 0$. In view of the above estimate, the inequality

$$\iint_{A \times A \setminus G_n} \xi \, dx' \, dx < \iint_{A \times A \setminus G_n} U^* \, dx' \, dx$$

is true for any n . Then the sequence defined by

$$r_n \doteq \iint_{A \times A \setminus G_n} (U^* - \xi) \, dx' \, dx$$

fulfills these requirements:

$$r_n > 0 \text{ and } r_n \nearrow r_0 \doteq \iint_{A \times A} (U^* - \xi) \, dx' \, dx > 0.$$

We write

$$\iint_{A \times A \setminus G_n} \xi \, dx' \, dx = \iint_{A \times A \setminus G_n} U^* \, dx' \, dx - r_n,$$

and we have

$$\iint_{A \times A \setminus G_n} \xi \, dx' \, dx < \iint_{A \times A \setminus G_n} U^* \, dx' \, dx - \frac{r_n}{2}$$

for any n . Thus, thanks to the convergence (40), the above inequality allows us to write

$$\iint_{A \times A \setminus G_n} \xi_\delta \, dx' \, dx < \iint_{A \times A \setminus G_n} U^* \, dx' \, dx - \frac{r_n}{2}$$

for any n and any $\delta \leq \delta_0$. If we pass to the limit when $n \rightarrow +\infty$, we get

$$\iint_{A \times A} \xi_\delta \, dx' \, dx \leq \iint_{A \times A} U^* \, dx' \, dx - \frac{r_0}{2}.$$

By taking now limits in $\delta \rightarrow 0$ and using (39), we have

$$\iint_{A \times A} U^* \, dx' \, dx \leq \iint_{A \times A} U^* \, dx' \, dx - \frac{r_0}{2}$$

which is a contradiction, and (41) has been proved.

Then, for each n and each measurable set $G \subset \Omega$, the biting convergence (40) enables us to write

$$\begin{aligned} \lim_{\delta \rightarrow 0} \iint_{G \times G} \xi_{\delta}(x', x) \, dx' dx &\geq \lim_{\delta \rightarrow 0} \iint_{G \times G \setminus G_n} \xi_{\delta}(x', x) \, dx' dx \\ &= \iint_{G \times G \setminus G_n} \xi(x', x) \, dx' dx. \end{aligned}$$

Finally, we take limits when $n \rightarrow +\infty$ and use (41) to obtain the desired estimate:

$$\begin{aligned} \lim_{\delta \rightarrow 0} \iint_{G \times G} \xi_{\delta}(x', x) \, dx' dx &\geq \lim_{n \rightarrow +\infty} \iint_{G \times G \setminus G_n} \xi(x', x) \, dx' dx \\ &= \iint_{G \times G} \xi(x', x) \, dx' dx \\ &\geq \iint_{G \times G} \frac{|\nabla u^*(x')|^2 + |\nabla u^*(x)|^2}{2|G|} dx' dx \\ &= \int_G |\nabla u^*(x)|^2 dx. \end{aligned}$$

Appendix B: $H_r(\delta_0)$ is Precompact in L^2

For this goal we shall use (43), an inequality due to Stein (see [9]), (44), an spectral result given in [17] (see also [33]), and a basic compactness result (Proposition B.1).

If $h \in H_r(\delta_0)$ then

$$B_0(h, h) = \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} k_{\delta_0}(|x' - x|) \frac{(h' - h)^2}{|x' - x|^2} dx' dx < r,$$

and $h = 0$ en $\Omega_{\delta_0} - \Omega$. We extend h by zero outside of Ω_{δ_0} and we compute the following integral:

$$\begin{aligned} &\int_{\Omega_{\delta_0+\delta}} \int_{\Omega_{\delta_0+\delta}} k_{\delta_0}(|x' - x|) \frac{(h' - h)^2}{|x' - x|^2} dx' dx \\ &= \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} + \int_{\Omega_{\delta_0+\delta} \setminus \Omega_{\delta_0}} \int_{\Omega_{\delta_0+\delta} \setminus \Omega_{\delta_0}} + 2 \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0+\delta} \setminus \Omega_{\delta_0}}. \end{aligned}$$

We note the second integral vanishes because $h(x) = h(x') = 0$ if $(x', x) \in (\Omega_{\delta_0+\delta} \setminus \Omega_{\delta_0}) \times (\Omega_{\delta_0+\delta} \setminus \Omega_{\delta_0})$.

The third one is zero too because h vanishes outside Ω

$$\begin{aligned} & \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0+\delta} \setminus \Omega_{\delta_0}} k_{\delta_0} (|x' - x|) \frac{(h(x') - h(x))^2}{|x' - x|^2} dx' dx \\ &= \int_{\Omega} \int_{\Omega_{\delta_0+\delta} \setminus \Omega_{\delta_0}} k_{\delta_0} (|x' - x|) \frac{(h(x))^2}{|x' - x|^2} dx' dx = 0 \end{aligned}$$

where the last equality is true thanks to $k_{\delta_0} (|x' - x|) = 0$ for $(x', x) \in (\Omega_{\delta_0+\delta} \setminus \Omega_{\delta_0}) \times (\Omega)$.

Therefore

$$B_0(h, h) = \int_{\Omega_{\delta_0+\delta}} \int_{\Omega_{\delta_0+\delta}} k_{\delta_0} (|x' - x|) \frac{(h' - h)^2}{|x' - x|^2} dx' dx < r. \quad (42)$$

Let $\varphi \in C_0^\infty(B(0, 1))$ be such that $\varphi \geq 0$ and $\int \varphi = 1$. For any $\delta > 0$ we define $\varphi_\delta(x) \doteq \frac{1}{\delta^N} \varphi\left(\frac{x}{\delta}\right)$ where $x \in \mathbb{R}^N$. At this point we can employ the following inequality

$$\int_{\Omega_{\delta_0+\delta}} \int_{\Omega_{\delta_0+\delta}} k_{\delta_0} (|x' - x|) \frac{(h' - h)^2}{|x' - x|^2} dx' dx \geq \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} k_{\delta_0} (|x' - x|) \frac{(h'_\delta - h_\delta)^2}{|x' - x|^2} dx' dx \quad (43)$$

where

$$h_\delta(x) = (\varphi_\delta * h)(x) \text{ for any } x \in \Omega_{\delta_0-\delta}, \delta_0 > \delta > 0.$$

Then h_δ is regular, $\text{supp } h \subset \overline{\Omega}$, $\text{supp } \varphi_\delta = B(0, \delta)$, and thus

$$\text{supp } h_\delta \subset \overline{\overline{\Omega} + \overline{B(0, \delta)}} = \overline{\Omega}_\delta \subset \Omega_{\delta_0-\delta} \subset \Omega_{\delta_0}.$$

We also know $h_\delta \rightarrow h$ in $L^2(\Omega_{\delta_0-\delta})$ if $\delta \rightarrow 0$ and, in particular, $h_\delta \rightarrow h$ strongly in $L^2(\Omega)$ if $\delta \rightarrow 0$. These facts clearly ensure $h_\delta \in X_0(\delta_0)$. We use Theorem 1.1 from [17] to write

$$B_0(h_\delta, h_\delta) = \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} k_{\delta_0} (|x' - x|) \frac{(h'_\delta - h_\delta)^2}{|x' - x|^2} dx' dx = \sum_k h_{\delta k}^2 \gamma_k^{nl}(\delta_0) \quad (44)$$

where $h_{\delta k} = \left(h_\delta, w_{\delta_0}^{(k)}\right)_{L^2(\Omega) \times L^2(\Omega)}$. If we take into account the limit

$$\lim_{\delta \rightarrow 0} h_{\delta k} = \lim_{\delta \rightarrow 0} \left(h_\delta, w_{\delta_0}^{(k)}\right)_{L^2(\Omega) \times L^2(\Omega)} = \left(h, w_{\delta_0}^{(k)}\right)_{L^2(\Omega) \times L^2(\Omega)} \doteq h_k,$$

the discussion from above, (42)–(44), gives rise to the inequality

$$r \geq \sum_k h_{\delta k}^2 \gamma_k^{nl}(\delta_0).$$

If we pass to the limit, we get

$$r \geq \lim_{\delta \rightarrow 0} \sum_k h_{\delta k}^2 \gamma_k^{nl}(\delta_0) \geq \sum_k h_k^2 \gamma_k^{nl}(\delta_0),$$

where $h_k = (h, w_{\delta_0}^{(k)})_{L^2(\Omega) \times L^2(\Omega)}$. We also recall the fact that $(w_{\delta_0}^{(k)})_k$ is an orthonormal basis in $L^2(\Omega)$ so that $\|h\|_{L^2(\Omega)}^2 = \sum_k h_k^2$.

We shall need this result (see [41]):

Proposition B.1 *Let ℓ^2 be the set of sequences $(u_n)_n$ such that $\sum u_n^2 < \infty$ and consider $(\rho_n)_n$ to be a sequence of positive numbers such that $\lim_{n \rightarrow +\infty} \rho_n = +\infty$. Let V be the space of sequences $(u_n)_n$ such that $\sum_n \rho_n |u_n|^2 < \infty$ and assume the space V is equipped with the scalar product $((u_n), (v_n)) = \sum \rho_n u_n v_n$. Then, V is a Hilbert space and $V \subset \ell^2$ with compact injection.*

Proposition B.2 *The set $\mathcal{H}_r(\delta_0)$ is precompact in $L^2(\Omega)$.*

Proof Let $(h_j)_j \subset \mathcal{H}_r(\delta_0)$ be any sequence. Then, for any j

$$r \geq \sum_k h_{jk}^2 \gamma_k^{nl}(\delta_0).$$

This estimate and Proposition B.1 guarantees $\{(h_{jk})_k\}_j$ converges in ℓ^2 if $j \rightarrow +\infty$, to a sequence $(\tilde{h}_k)_k$; in this way, if $\tilde{h}$ is the function from $L^2(\Omega)$ defined by means of the sequence $(\tilde{h}_k)_k$ (in the basis $(w_{\delta_0}^{(k)})_k$, $\tilde{h}_k = (\tilde{h}, (w_{\delta_0}^{(k)})_k)$), then we can claim

$$h_j \rightarrow \tilde{h} \text{ strongly in } L^2(\Omega).$$

Finally, since $(h_j)_j$ is uniformly bounded in L^∞ , we know there is a function $h \in L^\infty(\Omega)$ such that $h_j \rightharpoonup h$ weak-* in $L^\infty(\Omega)$. Hence $\tilde{h} = h$, and consequently $h \in \mathcal{H}_r(\delta_0)$. $\square$

References

1. Bates, P.W., Chmaj, A.: An integrodifferential model for phase transitions: stationary solutions in higher space dimensions. *J. Stat. Phys.* **95**(5–6), 1119–1139 (1999)
2. Silling, S.A.: Reformulation of elasticity theory for discontinuities and long-range forces. *J. Mech. Phys. Solids* **48**, 175–209 (2000)
3. Bazant, Z.P., Jirasek, M.: Nonlocal integral formulation of plasticity and damage: survey of progress. *J. Eng. Mech.* **128**, 1119–1149 (2002)
4. Bates, P.W., Zhao, G.: Existence, uniqueness and stability of stationary solution to a nonlocal evolution equation arising in population dispersal. *J. Math. Anal. Appl.* **332**(1), 428–440 (2007)
5. Berestycki, H., Nadin, G., Perthame, B., Ryzhik, L.: The non-local Fisher-KPP equation: travelling waves and steady states. *Nonlinearity* **22**(12), 2813–2844 (2009)
6. Bobaru, F., Duangpanya, M.: The peridynamic formulation for transient heat conduction. *Int. J. Heat Mass Transf.* **53**, 4047–4059 (2010)

7. Boulanger, J., Elbau, P., Pontow, C., Scherzer, O.: Non-local functionals for imaging. In: Fixed-point algorithms for inverse problems in science and engineering, vol. 49, pp. 31–154, Springer, New York (2011)
8. Alali, B., Lipton, R.: Multiscale dynamics of heterogeneous media in the peridynamic formulation. *J. Elast.* **106**(1), 71–103 (2012)
9. Ponce, A.C.: A new approach to Sobolev spaces and connection to Γ -convergence. *Cal. Var.* **19**, 229–255 (2004)
10. Chasseigne, E., Chaves, M., Rossi, J.D.: Asymptotic behavior for nonlocal diffusion equations. *J. Math. Pures Appl.* **86**(3), 271–291 (2006)
11. Andreu-Vaillo, F., Mazón, J.M., Rossi, J.D., Toledo-Melero, J.J.: Nonlocal Diffusion Problems, vol. 165. American mathematical society, Providence (2010)
12. Gunzburger, M., Lehoucq, R.B.: A nonlocal vector calculus with application to nonlocal boundary value problems. *Multiscale Model. simul.* **8**, 1581–1620 (2010)
13. Zhou, K., Du, Q.: Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions. *SIAM J. Numer. Anal.* **48**(5), 1759–1780 (2010)
14. Andreu, F., Mazón, J.M., Rossi, J.D., Toledo-Melero, J.J.: Local and nonlocal weighted p-Laplacian evolution equations with Neumann boundary conditions. *Publ. Mat.* **55**, 27–66 (2011)
15. Du, Q., Gunzburger, M.D., Lehoucq, R.B., Zhou, K.: Analysis and approximation of nonlocal diffusion problems with volume constraints. *SIAM Rev.* **54**(4), 667–696 (2012)
16. Du, Q., Gunzburger, M.D., Lehoucq, R.B., Zhou, K.: A nonlocal vector calculus, nonlocal volume-constrained problems and nonlocal balance laws. *Math. Models Methods Appl. Sci.* **23**, 493–540 (2013)
17. Andrés, F., Muñoz, J.: A type of nonlocal elliptic problem: existence and approximation through a Galerkin–Fourier Method. *SIAM J. Math. Anal.* **47**(1), 498–525 (2015)
18. Bellido, J.C., Mora-Corral, C., Pedregal, P.: Hyperelasticity as a Γ -limit of peridynamics when the horizon goes to zero. *Cal. Var.* (2015). doi:[10.1007/s00526-015-0839-9](https://doi.org/10.1007/s00526-015-0839-9)
19. Mengesha, T., Du, Q.: On the variational limit of a class of nonlocal functionals related to peridynamics. *Nonlinearity* **28**(11), 3999 (2015)
20. Allaire, G.: Shape Optimization by the Homogenization Method. Springer, New York (2002)
21. Cherkaev, A., Kohn, R. (eds.): Topics in Mathematical Modeling of Composite Materials. Birkhäuser, Boston (1997)
22. Delfour, M.C., Zolésio, J-P.: Shapes and geometries: metrics, analysis, differential calculus, and optimization, vol. 22. SIAM, Philadelphia (2011)
23. Dal, G.: Maso: An Introduction to Γ -convergence. Birkhäuser, Boston (1993)
24. Jikov, V.V., Kozlov, S.M., Oleinik, O.A.: Homogenization of Differential Operators and Integral Functionals. Springer, Heidelberg (1994)
25. Brasco, L., Parini, E., Squassina, M.: Stability for variational eigenvalues for the fractional p-laplacian. [arXiv:1503.04182v1](https://arxiv.org/abs/1503.04182v1) [math.AP] (2015)
26. D'Elia, M., Gunzburger, M.: Identification of the diffusion parameter in nonlocal steady diffusion problems, [arXiv:1310.2558v1](https://arxiv.org/abs/1310.2558v1) [math.OC] (2013)
27. D'Elia, M., Gunzburger, M.: Optimal distributed control of nonlocal steady diffusion problems. *SIAM J. Control Optim.* **52**(1), 243–273 (2014)
28. Teixeira, E.V., Teymurazyan, R.: Optimal design problems with fractional diffusions. *J. Lond. Math. Soc.* (2015). doi:[10.1112/jlms/jdv034](https://doi.org/10.1112/jlms/jdv034)
29. Andrés, F., Muñoz, J.: Nonlocal optimal design: a new perspective about the approximation of solutions in optimal design. *J. Math. Anal. Appl.* **429**, 288–310 (2015)
30. Aksoylu, B., Mengesha, T.: Results on nonlocal boundary value problems. *Numer. Funct. Anal. Optim.* **31**, 1301–1317 (2010)
31. Hinds, B., Radu, P.: Dirichlet's principle and wellposedness of solutions for a nonlocal p-Laplacian system. *Appl. Math. Comput.* **219**, 1411–1419 (2012)
32. Ponce, A.C.: An estimate in the spirit of Poincaré's inequality. *J. Eur. Math. Soc. (JEMS)* **6**, 1–15 (2004)
33. Servadei, R., Valdinoci, E.: Variational methods for non-local operators of elliptic type. *Discret. Contin. Dyn. Syst.* **33**(5), 2105–2137 (2013)
34. Bourgain, J., Brezis, H., Mironescu, P.: Another look at Sobolev spaces, another look at Sobolev spaces. In: J. L. Menaldi, E. Rofman, A. Sulem (eds.) *Optimal Control and Partial Differential Equations*, vol. 65 A, pp. 439–455. Bensoussan 60th birthday, IOS Press, Amsterdam (2001)

35. Dacorogna, B.: *Direct Methods in the Calculus of Variations*. Springer, New York (1989)
36. Hernández-Lerma, O., Lasserre, J.B.: Fatou's Lemma and Lebesgue's convergence theorem for measures. *J. Appl. Math. Stoch. Anal* **13**(2), 137–146 (2000)
37. Royden, H.L.: *Real Analysis*. Macmillan Publishing Company, New York (1988)
38. Cea, J., Malanowski, K.: An example of a Max–Min problem in partial differential equations. *SIAM J. Control* **8**(3), 305–316 (1970)
39. Kawohl, B., Stara, J., Wittum, G.: Analysis and numerical studies of a problem of shape design. *Arch. Ration. Mech. Anal.* **114**, 349–363 (1991)
40. Pedregal, P.: *Parametrized Measures and Variational Principles*. Birkhäuser, Boston (1997)
41. Brezis, H.: *Functional Analysis. Sobolev Spaces and Partial Differential Equations*. Springer, New York (2010)