A TYPE OF NONLOCAL ELLIPTIC PROBLEM: EXISTENCE AND APPROXIMATION THROUGH A GALERKIN–FOURIER METHOD*

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Abstract. The aim of this paper is to study a type of nonlocal elliptic equation whose format includes a kernel k and a design function h . We analyze how this equation is connected with the classical elliptic equation that includes h as diffusive term. On one hand, the spectrum of the nonlocal operator that defines the nonlocal equation is studied. Existence and unicity of solutions for the nonlocal equation are proved. On the other hand, the convergence of these solutions to the solution of the classical elliptic equation as the kernel k converges to a Dirac Delta is analyzed. This work is performed by using an spectral theorem on the nonlocal operator and by applying some specific compactness results. The kernel k is assumed to be radial. Dirichlet boundary conditions are assumed for the classical problem, whereas for the nonlocal equation a nonlocal boundary Dirichlet constraint must be defined.

Key words. nonlocality, peridynamics, approximation of partial differential equations, variational techniques

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1. Introduction.

1.1. Formulation of the problem. The present paper is concerned with the problem

$$(1.1) \quad (P_\delta) \equiv \begin{cases} L_\delta(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega^{nl} \doteq \Omega_\delta - \Omega, \end{cases}$$

where

$$L_\delta(u) = -2 \int_{B(x,\delta)} H(x',x) \frac{k_\delta(|x' - x|)}{|x' - x|^2} (u(x') - u(x)) dx', \quad x \in \Omega$$

and the following hold:

1. Ω is a smooth bounded domain in $\mathbb{R}^N$, and $\Omega_\delta = \Omega \cup \{\cup_{p \in \partial\Omega} B(p, \delta)\}$ ($B(x, r)$ is the notation for an open ball centered at $x \in \mathbb{R}^N$ and radius $r > 0$).
2. $f \in L^2(\Omega)$ and the design function H is given by

$$H(x', x) = \frac{h(x) + h(x')}{2}$$

with h in the space

$$L^\infty(\Omega_\delta; [h_{\min}, h_{\max}]) = \{h : h(x) \in [h_{\min}, h_{\max}] \text{ a.e. } x \in \Omega, h = 0 \text{ on } \Omega_\delta - \Omega\},$$

where the constants $0 < h_{\min} < h_{\max}$ are given.

3. The set $(k_\delta)_\delta$, $\delta > 0$, is a sequence of radial functions such that

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(a)

$$(1.2) \quad \frac{1}{N} \int_{B(0,\delta)} k_\delta(|s|) ds = 1,$$

(b)

$$(1.3) \quad \text{supp } k_\delta \subset B(0, \delta),$$

(c) $k_\delta \geq 0$ so that the following additional properties hold:

- i. The function $K_r : B(0, \delta) - \{0\} \rightarrow [0, +\infty)$ defined by the formula $K_r(z) \doteq \frac{k_\delta(|z|)}{|z|}$ is integrable, so that

$$(1.4) \quad K_r \in L^1(B(0, \delta)).$$

- ii. The function $K_s : B(0, \delta) - \{0\} \rightarrow [0, +\infty)$ defined by the formula $K_s(z) \doteq \frac{k_\delta(|z|)}{|z|^2}$ is assumed to be singular near the origin in the sense that

$$(1.5) \quad \lim_{\theta \rightarrow 0^+} \int_{B(0,\delta) - B(0,\theta)} K_s(z) dz = +\infty.$$

Problem (1.1) must be understood in a weak sense. To be precise we are going to specify the functional spaces we shall deal with. Throughout the paper $L_0^2(\Omega_\delta)$ is

$$L_0^2(\Omega_\delta) = \{u : \Omega_\delta \rightarrow \mathbb{R} : u \in L^2(\Omega), u = 0 \text{ in } \Omega_\delta - \Omega\},$$

and the space X is defined as

$$(1.6) \quad X = \{u \in L_0^2(\Omega_\delta) : B_h(u, u) < \infty\},$$

where

$$(1.7) \quad B_h(u, v) = \int_{\Omega_\delta} \int_{\Omega_\delta} H(x', x) \frac{k_\delta(|x' - x|)}{|x' - x|^2} (u(x') - u(x)) (v(x') - v(x)) dx' dx.$$

We also define the space X_0 as

$$X_0 = \overline{C_{co}^\infty(\Omega_\delta)},$$

where

$$C_{co}^\infty(\Omega_\delta) = \{f : \Omega_\delta \rightarrow \mathbb{R} : f \in C_c^\infty(\Omega) \text{ and } f = 0 \text{ in } \Omega_\delta - \Omega\} \subset X$$

and $\overline{C_{co}^\infty(\Omega_\delta)}$ is the closure with respect to the norm $\|\cdot\|$ given in X via the quadratic form $B_h(\cdot, \cdot)$, that is,

$$X_0 = \left\{ v \in X : \text{there is } (v_j) \subset C_{co}^\infty(\Omega_\delta) \text{ such that } \lim_j B_h(v_j - v, v_j - v) = 0 \right\}.$$

DEFINITION 1. *It is said that $u \in X_0$ is a solution for (P_δ) , the problem (1.1), if the equality*

$$(1.8) \quad B_h(u, v) = (f, v)_{L^2(\Omega) \times L^2(\Omega)}$$

holds for any $v \in X_0$.

Remark 1.1. For clarity of the integral expressions, we will often use the notation $u' = u(x')$, $u = u(x)$ and $H = H(x', x)$. Also, for a given domain ω its measure is written like $|\omega|$. The bilinear form $(\cdot, \cdot)_{L^2(\omega) \times L^2(\omega)}$ stands for the inner product in $L^2(\omega)$, that is, $(f, v)_{L^2(\omega) \times L^2(\omega)} \doteq \int_\omega f(x)v(x)dx$.

1.2. Results and organization. In a summarized way, the aim of the present research is described as follows:

1. The first goal is to give the proof of an spectral theorem for the nonlocal operator L_δ (section 3). In section 2, and under the specific hypotheses (1.4) and (1.5), we shall express the bilinear form B_h with $h = 1$ in terms of series. We use that to finally achieve a certain compactness result, which will be essential for the proof of the spectral theorem. Apart from that, we gather in Remark 3.1 some other technical results derived from section 2 that will be of fundamental importance in section 3.
2. The second goal is the existence and unicity of solution for the problem (1.1).
3. Besides, the third goal concerns the analysis of the problem when the *horizon* parameter δ tends to zero. We pose whether the sequence of solutions $(u_\delta)_\delta$ provides an approximations for the solutions of the classic model (P) when $\delta \rightarrow 0$. The classic model is the elliptic equation under Dirichlet boundary condition:

$$(1.9) \quad (P) \equiv \begin{cases} -\operatorname{div}_x (h(x) \nabla u(x)) = f(x) \text{ in } \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

According to those objectives, the results we obtain are stated and commented in what follows.

The first theorem establishes the existence of eigenvalues and eigenfunctions for the nonlocal operator.

THEOREM 1.1 (spectral theorem). *There is a sequence of functions $(w_\delta^{(k)})_k \subset X_0$ and a sequence of positive real numbers $(\gamma_k^{nl})_k$, $\gamma_k^{nl} = \gamma_k^{nl}(\delta)$ such that*

$$\begin{aligned} & \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x' - x|) H(x', x) \frac{(w_\delta^{(k)}(x') - w_\delta^{(k)}(x))(v(x') - v(x))}{|x' - x|^2} dx dx' \\ & = \gamma_k^{nl} \int_{\Omega} w_\delta^{(k)}(x) v(x) dx \end{aligned}$$

or simply

$$B_h(w_\delta^{(k)}, v) = \gamma_k^{nl} (w_\delta^{(k)}, v)_{L^2(\Omega) \times L^2(\Omega)}$$

for any $v \in X_0$. Moreover, the following hold:

1. The sequence $\gamma_k^{nl}(\delta)$ is nondecreasing, and $\gamma_k^{nl}(\delta) \rightarrow +\infty$ if $k \rightarrow +\infty$.
2. $\{w_\delta^{(k)}\}$ is an orthonormal basis in $L_0^2(\Omega_\delta)$ and orthogonal basis in X_0 .
- 3.

$$\gamma_k^{nl}(\delta) = B_h(w_\delta^{(k)}, w_\delta^{(k)}) = \min_{u \in X^{(k)} : \|u\|_{L_0^2(\Omega_\delta)} = 1} B_h(u, u),$$

where

$$X^{(k)} = \left\{ v \in X_0 : B_h(v, w_\delta^{(j)}) = 0, j = 1, 2, \dots, k-1 \right\}$$

for $k \geq 2$ and $X^{(1)} = X_0$.

Somehow the task we have performed here is complementary to the work done by Zhou and Du in the papers [36] and [37]. Factually, some essential clues for our research come from those works. In the present manuscript, we study the spectrum of a nonlocal operator in a case where the model includes a design function h , the

geometry of Ω is smooth, and no assumption about the behavior of the admissible functions on the boundary $\partial\Omega$ is made, except for the nonlocal boundary Dirichlet condition ($u = 0$ in $\partial\Omega^{nl}$). We must emphasize that in our research, a meaningful part of the work consists on finding a concrete representation of the nonlocal functional by means of series involving the corresponding nonlocal eigenvalues (see subsection 2.2). Once this representation is proved, we face the proof of the spectral theorem.

The spectral analysis performed in this work is quite classical and very similar to the one developed in [33]. Apart from some minor details on the hypotheses, the great difference basically stems from the way of finding the required compactness needed for the existence of eigenfunctions and eigenvalues. We do not assume any explicit lower bound for our kernel. We assume nonnegative, radial kernels for which the condition of integrability (1.4) and the divergence condition (1.5) are fulfilled.

Whereas some authors consider hypotheses for which the integrability of the kernel holds and thereby the nonlocal problem is formulated in the underlying L^2 space (see [14, Formula (7.9)]), some others assume singular behavior for the kernel in order to achieve a problem formulated within the corresponding fractional Sobolev space (see, for instance, [14, Formula 7.8] and [33]). These tasks have been usually carried out in order to get the desired compactness property, which is used to search the eigenfunctions and the eigenvectors of the nonlocal operator. Our investigation does not treat the different ways this meaningful property is accomplished; rather, our framework provides a type of kernel, not so integrable as is the case in L^2 and not so singular as is the case of the fractional Sobolev space either. To be more concrete, and writing the kernel as $\gamma(x, y) = \frac{k(|x-y|)}{|x-y|^2}$, a kind of γ our research includes is $\gamma(x, y) = \frac{1}{|x-y|^\beta}$ with $N \leq \beta < N + 1$. Notice that the prototype kernel $\gamma(x, y) = \frac{1}{|x-y|^{N+2s}}$ is included in our framework if $0 < s < 1/2$. In order to show the generality of our conditions, a kind of kernel for which all the requirements of our framework are fulfilled is $\gamma(x, y) = \frac{\exp(-|x-y|)}{|x-y|^N}$.

The analysis we perform in section 2 is a first step in order to prove Theorem 1.1 in section 3. It focuses on writing the quadratic functional $B_h(u, u)$ with $h = 1$ by means of series involving the Fourier coefficients of u in the appropriate basis of eigenfunctions $(e_k)_k$ (see Theorem 2.1). It is in this step of our research where (1.4) plays a crucial role. We shall prove the expansion

$$(1.10) \quad B(u, v) = \sum_{k=1}^{\infty} \lambda_k^{nl} u_k v_k,$$

where the u_k and v_k are the Fourier coefficients of admissible functions u and v , respectively (see [36]). $(\lambda_k^{nl})_k$ is a sequence of nonnegative numbers fulfilling $\lim_{k \rightarrow +\infty} \lambda_k^{nl} = +\infty$. The divergence of this limit comes from the assumption (1.5), and it will be the essential ingredient to ensure compactness (see Proposition 6.1 in the appendix). For instance, if $\gamma(x, y) \geq \frac{1}{|x-y|^{N+2s}}$ with $1/2 \leq s < 1$, then, even though the above expansion (1.10) cannot be ensured, Theorem 1.1 remains valid once we assume suitable hypotheses on the kernel (see Remark 3.2). Moreover, in either case, Theorems 1.2, 1.3, and 1.4 can be independently proved once the thesis of Theorem 1.1 is proved.

The next goal is the existence of a solution for the nonlocal problem (P_δ) . The existence of a basis of eigenfunctions is the essential tool for the proof of existence. This proof is done by using the Galerkin–Fourier method (see [22] or [16]).

THEOREM 1.2. *For any $\delta > 0$, there exists a unique solution u_δ to the problem (P_δ) .*

We shall deal with existence under the point of view of approximation. There are works where this issue is proved under quite general assumptions and via the direct methods of the calculus of variations (see, for instance, [20] or [1]). We recall that the objective we pursue is more specific; our aim clearly hinges on finding expansions for the solutions of the nonlocal problem that in the limit, when the horizon tends to zero, approximate the solution of the local problem (P) .

The writing of solutions as series provides an easy way to prove the convergence to the solution of the local problem. In doing this, a compactness result by Ponce will play a crucial role (see [25]). For an intuitive introduction on the approximation of the problem (P) by (P_δ) , we refer to [10], [3], [19], or [23].

The first step in this analysis consists of studying the asymptotic of the spectrum. Let $\cup_{k=1}^\infty \cup_{i=1}^{l_k} \psi_{ik}$ be an orthonormal basis in $L^2(\Omega)$, and orthogonal in $H_0^1(\Omega)$, of eigenfunctions for the local problem (P) on Ω with spectrum $(\gamma_k)_k$ such that γ_k is the corresponding eigenvalue of the eigenspace $\mathcal{L}\{\cup_{i=1}^{l_k} \psi_{ik}\}$. The notation is the same as for the nonlocal problem (1.8): thanks to Theorem 1.1 we know $\cup_{k=1}^\infty \cup_{i=1}^{n_k} w_{i\delta}^{(k)}$ is a basis of nonlocal eigenvectors whose spectrum is $(\gamma_k^{nl}(\delta))_k$, where $\gamma_k^{nl}(\delta)$ is the eigenvalue whose eigenspace is $\mathcal{L}\{\cup_{i=1}^{n_k} w_{i\delta}^{(k)}\}$. Under the preceding circumstances, we shall be able to derive the following meaningful result.

THEOREM 1.3.

1. Concerning the first eigenvalue and the first eigenfunction ($l_1 = 1$), there is a subsequence of δ 's we still denote by δ such that

$$\lim_{\delta \rightarrow 0} \gamma_1^{nl}(\delta) = \gamma_1$$

and

$$\lim_{\delta \rightarrow 0} w_\delta^{(1)} = \psi_1 \text{ strong in } L^2(\Omega).$$

2. For each $k \geq 2$, we have $n_k = l_k$, and there is a subsequence of δ 's (still denoted by δ) and a set of functions $\cup_{i=1}^{n_k} w_{i\delta}^{(k)}$ in $H_0^1(\Omega)$ such that

$$\lim_{\delta \rightarrow 0} \gamma_k^{nl}(\delta) = \gamma_k$$

and

$$\lim_{\delta \rightarrow 0} w_{i\delta}^{(k)} = w_i^{(k)} \text{ strong in } L^2(\Omega), \quad i = 1, \dots, n_k,$$

where $w_i^{(k)}$ are eigenfunctions for the local problem with eigenvalue γ_k , $i = 1, \dots, n_k$. Moreover, the set $\cup_{k=1}^\infty \cup_{i=1}^{n_k} w_i^{(k)}$ is a basis in $L^2(\Omega)$ (and also in $H_0^1(\Omega)$).

Our chief purpose is the following.

THEOREM 1.4. If $(u_\delta)_\delta$ is the sequence of solutions of (P_δ) , then the following hold.

1. There exists $u \in H_0^1(\Omega)$ and subsequence of $(u_\delta)_\delta$ (which is going to be still denoted by $(u_\delta)_\delta$) such that

$$u_\delta \rightarrow u \text{ strong in } L^2(\Omega)$$

and

$$u_\delta \rightarrow u \text{ strong in } X.$$

2. $u \in H_0^1(\Omega)$ is the solution of the problem (P) .

The type of functionals we study here have appeared in several contexts of research: in nonlocal models in ferromagnetism [27], or in diffusion problems (see [4] and references therein), as regularization of nonconvex variational principles [7], [24] and in image analysis [18], [9]. Perhaps the field where this kind of functional has attracted much more attention is in the nonlocal elasticity setting. (The first works in this area were given by Silling [29], [32].) The nonlocal calculus exhibited there is an alternative to standard approaches for circumventing the technicalities associated with a lack of sufficient regularity. Some outstanding works from this area are [30], [34], [13], [31], [35], and [2].

In the specific issue related to the formulation and existence of solutions of nonlocal problems, we must mention among others these works: [36], [1], [19], [20], [15], [8], [6]. Relative to the approximation to the local problem, the papers [19], [14], [36], and [37] must be highlighted. In particular, we must mention [36] and [37], where similar results to Theorem 1.4 have been established. They consider that $h = 1$, Ω is an interval if $N = 1$, and a cube if $N = 2$ (in [36]) or $\Omega = \mathbb{R}^N$ (in [37]). Also, the collection of works by Rossi and co-authors is a very interesting, comprehensive, and wide perspective from which to face nonlocal problems (see, for instance, [3] or [5]). The paper [17] and references therein are interesting sources where the asymptotic properties of nonlocal eigenvalues are analyzed.

Some of the deepest tools and main ideas for the carrying out of the present manuscript come from [25]. This work belongs to a collection of papers regarding to compactness results in a nonlocal setting and to the nonlocal approximation of Sobolev spaces (consult the seminal papers [10], [11], and [26]).

As we have already commented, a significance work related to the spectral theory of nonlocal operators that in some aspects is similar to the present paper is [33].

The organization of the paper is as follows: section 2 is devoted to analyze the nonlocal operator B_h with $h \equiv 1$. We prove Theorem 2.1. This result is of great importance in order to face the proof of Theorem 1.1 in section 3. Section 4 contains the proof of Theorem 1.2. Finally, the proofs of Theorems 1.3 and 1.4 are given in section 5. We gather some key tools and computations in the appendix (section 6).

2. Previous remarks.

2.1. A remarkable digression. In order to understand (1.8), it is suitable to point out the relation between B_h and L_δ . If $(u, v) \in C^\infty(\Omega_\delta) \times C_{co}^\infty(\Omega_\delta)$, then according to (1.4)

$$\frac{1}{2} H \frac{k_\delta(|x' - x|)}{|x' - x|^2} (u' - u) (v' + v) \in L^1(\Omega_\delta \times \Omega_\delta).$$

Thus, due to the symmetry of the domain of integration, we have

$$\begin{aligned} \int_{\Omega_\delta} \int_{\Omega_\delta} H \frac{k_\delta(|x' - x|)}{|x' - x|^2} (u' - u) (v' - v) dx' dx \\ = -2 \int_{\Omega_\delta} \left(\int_{\Omega_\delta} H \frac{k_\delta(|x' - x|)}{|x' - x|^2} (u' - u) dx' \right) v dx \end{aligned}$$

and consequently

(2.1)

$$B_h(u, v) = -2 \int_{\Omega} \left(\int_{B(x, \delta)} H \frac{k_\delta(|x' - x|)}{|x' - x|^2} (u' - u) dx' \right) v dx = (L_\delta(u), v)_{L^2(\Omega) \times L^2(\Omega)}.$$

Namely, $L_\delta(u)$, defined as a linear L^2 -operator, is identical to $B_h(u, \cdot)$ if u is assumed to be enough regular.

2.2. Analysis of B_h with $h = 1$. To understand the model described above, we shall analyze a particular kind of eigenfunctions and eigenvalues of B_h with $h \equiv 1$. In this case, we shall denote the bilinear form $B \doteq B_1$.

We fix $d \geq 0$. The existence of an orthonormal Hilbert basis (φ_n^d) in $L^2(\Omega_d)$ (also an orthogonal Hilbert basis in $H_0^1(\Omega_d)$) and the corresponding sequence of real positive numbers $(\lambda_n(d))_{n=1}^\infty$, such that $\lambda_n(d) \rightarrow +\infty$ if $n \rightarrow +\infty$, $\varphi_n^d \in H_0^1(\Omega_d) \cap C^\infty(\overline{\Omega_d})$, and $-\Delta \varphi_n^d = \lambda_n(d) \varphi_n^d$ in Ω_d , is well known.

It is also well known that the above statement can be extended for more general elliptic problems: there exists a Hilbert basis $(\psi_n^d)_n$ in $H^1(\Omega_d)$ and there exists a sequence $(\gamma_n^d)_n$ of positive real numbers with $\gamma_n^d \rightarrow +\infty$ such that $\psi_n^d \in H_0^1(\Omega_d)$ and

$$(2.2) \quad -\operatorname{div}(h \nabla \psi_n^d) = \gamma_n^d \psi_n^d$$

in $H_0^1(\Omega_d)$ (see [12] or [16] and the references therein).

We now give the most important preliminary; instead of considering $(\varphi_n^d)_n$ or $(\psi_n^d)_n$, we are going to take the sequence of eigenfunctions $(e_n)_n$ of $-\Delta$ operator in a hypercube $Q \subset \mathbb{R}^N$ centered at the origin and such that $\Omega \subset Q$: they fulfill the equation

$$(2.3) \quad -\Delta e_n = \lambda_n e_n \text{ in } H_0^1(Q),$$

$e_n \in H_0^1(Q) \cap C^\infty(Q)$, they are analytic in Q , and the sequence $(e_n)_n$ constitutes a Hilbert basis in $L^2(Q)$ (in particular it is in $L^2(\Omega)$). Explicitly, we choose $Q = \Pi_{i=1}^N [a_i, b_i]$. Then the eigenfunctions have the format

$$(2.4) \quad e_n(x) = \frac{1}{c_n} \Pi_{i=1}^N \sin \left(k_i \pi \frac{x_i - a_i}{b_i - a_i} \right),$$

where $x = (x_1, \dots, x_N)$, $k_i \in \mathbb{N}$, $i = 1, \dots, N$, $c_n = \Pi_{i=1}^N \left(\frac{b_i - a_i}{2} \right)^{\frac{1}{2}}$ and the underlying eigenvalues are $\lambda_n = \sum_{i=1}^N \left(\frac{k_i \pi}{b_i - a_i} \right)^2$. These functions are obviously analytic with spectral radius $R = +\infty$.

THEOREM 2.1. *Under the previous circumstances*

$$(2.5) \quad L_\delta(e_n) = \lambda_n^{nl}(\delta) e_n$$

and

$$(2.6) \quad B(e_n, v) = \lambda_n^{nl}(\delta) (e_n, v)_{L^2(\Omega) \times L^2(\Omega)}$$

for any $v \in X_0$ and any $n \in \mathbb{N}$, where

$$(2.7) \quad \lambda_n^{nl}(\delta) = 2 \int_{B(0, \delta)} \frac{k_\delta(|s|) \left(1 - \cos \left(\sqrt{\frac{\lambda_n}{N}} (s \cdot \mathbf{1}) \right) \right)}{|s|^2} ds$$

and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^N$.

Moreover, if $u_k = (u, e_k)_{L^2(\Omega) \times L^2(\Omega)}$ and $v_k = (v, e_k)_{L^2(\Omega) \times L^2(\Omega)}$, then

$$(2.8) \quad B(u, v) = \sum_{k=1}^{\infty} \lambda_k^{nl} u_k v_k$$

for any $u, v \in X_0$.

Remark 2.1. Assume $N = 1$ and consider the sequence of eigenfunction for the Laplacian $(\sin nx)_n$. In general, the functions $\sin nx$, $n \in \mathbb{N}$, are not eigenfunctions for the operator L_δ . It is easy to be convinced of this assertion because simple calculations give

$$\begin{aligned} L_\delta (\sin nx) &= -2 \int_{-\delta}^{\delta} \frac{k_\delta(|s|)}{|s|^2} (\sin(n(x+s)) - \sin(nx)) ds \\ &= 2 \int_{-\delta}^{\delta} \frac{k_\delta(|s|)}{|s|^2} (1 - \cos ns) ds \sin nx - 2 \int_{-\delta}^{\delta} \frac{k_\delta(|s|)}{|s|^2} \sin ns ds \cos nx. \end{aligned}$$

Then, we note the last integral is singular and does not vanish, unless the function $\frac{k_\delta(|s|)}{|s|} \in L^1(-\delta, \delta)$. In such a case, the corresponding eigenvalue would be $\lambda_n = 2 \int_{-\delta}^{\delta} \frac{k_\delta(|s|)}{|s|^2} (1 - \cos ns) ds$. Thus, if we do not assume some sort of integrability condition, the action of L_δ is not well defined and (2.5) may fail.¹

In order to prove Theorem 2.1, we shall analyze the right and left hand of (2.6). Concerning the left hand, we have the following.

LEMMA 2.1. *Let $e_r^{(j)}$ be the corresponding tensors to the differential form of order j for the function e_r , $r = 1, 2, \dots$. If we define the series*

$$(2.9) \quad f^r(x', x) = \sum_{j=1}^{\infty} f_j^r(x', x),$$

where $f_j^r(x', x) \doteq \frac{k_\delta(|x'-x|)}{|x'-x|^2} \frac{e_r^{(j)}(x)}{j!} \cdot (x' - x)^j$, then $f^r \in L^1(\Omega_\delta \times \Omega_\delta)$ and

$$(2.10) \quad B(e_r, v) = -2 \sum_{j=1}^{\infty} \int_{\Omega} \int_{B(x, \delta)} f_j^r(x', x) v(x) dx' dx$$

for any $v \in C_{co}^\infty(\Omega_\delta)$.

Proof. Since e_r is analytic in Ω_δ , we have the expansion

$$f^r(x', x) = \sum_{j=1}^{\infty} f_j^r(x', x) = \frac{k_\delta(|x' - x|)}{|x' - x|^2} (e'_r - e_r)$$

if $x' \neq x$. Therefore, by using (2.1) we can write

$$(2.11) \quad B(e_r, v) = -2 \int_{\Omega} \int_{B(x, \delta)} \sum_{j=1}^{\infty} f_j^r(x', x) v(x) dx' dx.$$

Consequently, the aim is to see whether we can interchange the series and the integrals in the right hand of the above equality. The proof is based on a classical results from real analysis (see [28]): in order to deduce our thesis, it is enough to prove

$$I \doteq \sum_{j=1}^{\infty} \int_{\Omega} \int_{B(x, \delta)} |f_j^r(x', x) v(x)| dx' dx < \infty.$$

¹This functional should be defined in a suitable sense (by means of the principal value of the integral or as an specific distribution).

Clearly, there is a positive constant C (which depends on δ) such that

$$\int_{\Omega} \int_{B(x, \delta)} |f_j^r(x', x) v(x)| dx' dx \leq C \frac{\delta^{j-1}}{j!} \|e_r^{(j)}(x)\|_{L^2(\Omega_\delta)}.$$

We employ now a regularity result on elliptic equations which, in the case of the eigenfunction for the Laplacian operator (2.3), ensures the existence of a positive constant $C_0 = C_0(\Omega_\delta)$ such that

$$(2.12) \quad \|e_r^{(j)}(x)\|_{L^2(\Omega)}^2 \leq C_0^j (\lambda_r + 1)^j$$

for any j (see [16]). Thus

$$I \leq \frac{C}{\delta} \sum_{j=1}^{\infty} \frac{(\sqrt[2]{\lambda_r + 1} \delta)^j}{j!} < +\infty,$$

and this finishes the proof. $\square$

The analysis of the right hand of (2.6) is about the properties of the values that ultimately will turn out to play a central role in the remainder of this section.

PROPOSITION 2.1 (see [36]). *Under the hypotheses (1.2)–(1.4), and for any $\delta \leq \delta_0$ (δ_0 fixed), we get*

1. $\inf_{n \geq 1} \{\lambda_n^{nl}\} > 0$,
2. $\lambda_n^{nl} \leq \lambda_n$ and $\lim_{\delta \rightarrow 0} \lambda_n^{nl} = \lambda_n$,
3. $\lim_{n \rightarrow +\infty} \lambda_n^{nl} = +\infty$.

Proof.

1. Clearly $\lambda_n^{nl} > 0$ and

$$\lambda_n^{nl} \geq \frac{1}{\delta^2} \int_{B(0, \delta)} k_\delta(|s|) \left(1 - \cos\left(\sqrt[2]{\lambda_n}(s \cdot \mathbf{e})\right)\right) ds,$$

where $\mathbf{e} \doteq \frac{1}{\sqrt[2]{N}}$. We note the function $\beta : s \rightarrow 1 - \cos(s \cdot \mathbf{e})$ is in $L^\infty(\mathbb{R}^N)$ and

$$\lim_{R \rightarrow +\infty} \frac{\int_0^R \beta(r\omega) dr}{R} = 1.$$

Since $\frac{1}{N} \int_{B(0, \delta)} k_\delta(|s|) ds = 1$, the generalized Riemann–Lebesgue lemma (see Proposition 6.2) allows us to set the equality

$$\lim_{n \rightarrow +\infty} \int_{B(0, \delta)} k_\delta(|s|) \left(1 - \cos\left(\sqrt[2]{\lambda_n}(s \cdot \mathbf{e})\right)\right) ds = N.$$

We have just obtained $\lim_{n \rightarrow +\infty} \lambda_n^{nl} > \frac{N}{\delta_0^2}$. This states the existence of a positive constant C (independent of δ) such that $\lambda_n^{nl} \geq C$ for any n .

2. Concerning (2), we employ the fact $1 - \cos x = (x^2/2) \cos(\eta(x))$ for some $\eta(x) \in [0, x]$, the mean value theorem for integrals, and a change of variables ($s = \rho\omega$) to get

$$\begin{aligned} \lambda_n^{nl} &= \lambda_n \cos\left(\sqrt[2]{\lambda_n} \eta(s_0) \cdot \mathbf{e}\right) \int_0^\delta k_\delta(\rho) \rho^{N-1} \int_{S^{N-1}} (\omega \cdot \mathbf{e})^2 d\sigma(\omega) d\rho \\ &= \lambda_n \cos\left(\sqrt[2]{\lambda_n} \eta(s_0) \cdot \mathbf{e}\right), \end{aligned}$$

where $\eta(s_0) \in B(0, \delta)$ and $\eta(s_0) \cdot \mathbf{1} \rightarrow 0$ if $\delta \rightarrow 0$. For the sake of clarity, in the

above computations we have employed the identities $\frac{1}{N}|S^{N-1}|\int_0^\delta k_\delta(\rho)\rho^{N-1}d\rho = 1$ and $\frac{1}{|S^{N-1}|}\int_{S^{N-1}}(\omega \cdot \mathbf{e})^2 d\sigma(\omega) = \frac{1}{N}$. (Here, σ stands for the $N-1$ -dimensional Hausdorff measure on S^{N-1} , the unit sphere in $\mathbb{R}^N$.)

3. Clearly, for θ small enough

$$\int_{B(0,\epsilon)-B(0,\theta)} \frac{k_\delta(|s|)}{|s|^2} ds < \infty.$$

Also, it is obvious that

$$\lambda_n^{nl} \geq \int_{B(0,\epsilon)-B(0,\theta)} \frac{k_\delta(|s|)}{|s|^2} \left(1 - \cos \sqrt{\lambda_n}(s \cdot \mathbf{e})\right) ds,$$

and thereby the generalized Riemann–Lebesgue lemma guarantees the bound

$$\begin{aligned} \lim_{n \rightarrow +\infty} \lambda_n^{nl} &\geq \lim_{n \rightarrow +\infty} \int_{B(0,\epsilon)-B(0,\theta)} \frac{k_\delta(|s|)}{|s|^2} \left(1 - \cos \sqrt{\lambda_n}(s \cdot \mathbf{e})\right) ds \\ &= \int_{B(0,\epsilon)-B(0,\theta)} \frac{k_\delta(|s|)}{|s|^2} ds. \end{aligned}$$

If we take limits when $\theta \rightarrow 0^+$ in the inequality from above and use (1.5), we get the desired thesis. $\square$

Proof (first part of the proof of Theorem 2.1). In this part, we limit the proof to the case where u and v belongs to $C_{co}^\infty(\Omega_\delta)$.

For the sake of clarity, we shall give the proof for $N = 2$. Apart from a huge and tedious manipulation, the procedure for high dimension is identical to the one exhibited here for the two-dimensional case.

We split the proof into two steps.

Step 1. The aim now is to show

$$(2.13) \quad L_\delta(u) = \lambda_r^{nl} u$$

and

$$(2.14) \quad B(u, v) = \lambda_r^{nl}(u, v)_{L^2(\Omega) \times L^2(\Omega)},$$

where v is any function from $C_{co}^\infty(\Omega_\delta)$, $u = e_r$ is any eigenfunction for the $-\Delta$ operator, and λ_r^{nl} has been defined in (2.7). We use the following notation in order to work with the differential forms of arbitrary orders: if $z = (z_1, z_2)$, we have

$$(2.15) \quad u^{(i)}(x) \cdot \left(z \otimes \cdots \otimes z\right)^i = \left(\frac{\partial u}{\partial x_1} z_1 + \frac{\partial u}{\partial x_2} z_2\right)^i = \sum_{m=0}^i \binom{i}{m} \frac{\partial^i u}{\partial x_1^{i-m} \partial x_2^m} z_1^{i-m} z_2^m.$$

Also, we are going to make use of the following chain of identities whose proof is given in section 6: it holds that

$$\begin{aligned} (2.16) \quad &\int_{B(0,\delta)} \frac{k_\delta(|z|)}{|z|^2} \left(\frac{\partial u(x)}{\partial x_1} z_1 + \frac{\partial u(x)}{\partial x_2} z_2\right)^{2n} dz \\ &= \Delta^n u(x) \int_{B(0,\delta)} \frac{k_\delta(|z|)}{|z|^2} (z_1)^{2n} dz \\ &= (-1)^n u(x) \int_{B(0,\delta)} \frac{k_\delta(|z|)}{|z|^2} \left(\sqrt{\frac{\lambda}{2}}(z_1 + z_2)\right)^{2n} dz, \end{aligned}$$

where λ is the eigenvalues of u and Δ^n is the n -Laplacian.

Thanks to (2.10) and (2.15),

$$B(u, v) = \lim_{M \rightarrow \infty} \sum_{i=1}^M I_i,$$

where

$$I_i = -2 \int_{\Omega} \int_{B(0, \delta)} \frac{k_{\delta}(|z|)}{i! |z|^2} \left(\frac{\partial u(x)}{\partial x_1} z_1 + \frac{\partial u(x)}{\partial x_2} z_2 \right)^i v(x) dz dx.$$

If i is odd, then $I_i = 0$ since for $0 \leq m \leq i$ it is clear that

$$\int_{B(0, \delta)} \frac{k_{\delta}(|z|)}{|z|^2} z_1^{i-m} z_2^m dz = 0.$$

It remains to compute I_i with $i = 2n$ and $n \in \mathbb{N}$, but this is

$$I_{2n} = -2 \int_{\Omega} \int_{B(0, \delta)} \frac{k_{\delta}(|z|)}{|z|^2} \frac{1}{(2n)!} \left(\frac{\partial u}{\partial x_1} z_1 + \frac{\partial u}{\partial x_2} z_2 \right)^{2n} v(x) dz dx.$$

After applying (2.16), we have

$$\begin{aligned} & \sum_{n=1}^M I_{2n} \\ &= -2 \sum_{n=1}^M \int_{\Omega} (-1)^n u(x) v(x) \int_{B(0, \delta)} \frac{k_{\delta}(|z|)}{(2n)! |z|^2} \left(\sqrt{\frac{\lambda}{2}} (z_1 + z_2) \right)^{2n} dz dx \\ &= 2 \int_{B(0, \delta)} \frac{k_{\delta}(|z|)}{|z|^2} \sum_{i=1}^M \frac{(-1)^{n+1}}{(2n)!} \left(\sqrt{\frac{\lambda}{2}} (z_1 + z_2) \right)^{2n} dz \int_{\Omega} u(x) v(x) dx. \end{aligned}$$

Then, if we pass to the limit when $M \rightarrow +\infty$, we get the formula (2.14):

$$B(u, v) = 2 \int_{B(0, \delta)} \frac{k_{\delta}(|z|)}{|z|^2} \left(1 - \cos \sqrt{\frac{\lambda}{2}} (z_1 + z_2) \right) dz \int_{\Omega} u(x) v(x) dx.$$

By suppressing the integral on Ω and v in all the integrals from above, we easily check that (2.13) has been proved.

Step 2. Assume $v, u \in C_{co}^{\infty}(\Omega_{\delta})$. Then $v \in H_0^1(\Omega)$, and this function can be approximated in the norm of $H^1(\Omega_{\delta})$ (or in $H^1(Q)$ if we assume v is extended by zero in $Q - \Omega_{\delta}$) by a sequence of functions from the linear space generated by the eigenfunctions e_s . Thus, we can write

$$v(x) = \sum_{s=1}^{\infty} (v, e_s) e_s = \sum_{s=1}^N (v, e_s) e_s(x) + R_{N+1}(x),$$

where

$$(v, e_s) = \int_Q v(x) e_s(x) dx = \int_{\Omega} v(x) e_s(x) dx = (v, e_s)_{L^2(\Omega) \times L^2(\Omega)},$$

and $\|R_{N+1}\|_{H^1(\Omega_\delta)} \rightarrow 0$ if $N \rightarrow +\infty$. The action of B upon the pair (v, u) is

$$B(v, u) = \sum_{s=1}^N (v, e_s) B(e_s, u) + B(R_{N+1}, u).$$

By Holder inequality, we have

$$|B(R_{N+1}, u)| \leq (B(u, u))^{1/2} (B(R_{N+1}, R_{N+1}))^{1/2},$$

and hence, inequality (6.1) from the appendix serves to state the bound

$$B(R_{N+1}, R_{N+1}) \leq C \int_{\Omega_\delta} |\nabla R_{N+1}|^2 dx.$$

Since the last integral converges to 0 if $N \rightarrow +\infty$ we can apply the preceding step to get

$$B(v, u) = \lim_{N \rightarrow \infty} \sum_{s=1}^N (v, e_s) B(e_s, u) = \sum_{s=1}^{\infty} \lambda_s^{nl} (v, e_s) (u, e_s).$$

In particular

$$(2.17) \quad B(v, v) = \sum_{s=1}^{\infty} \lambda_s^{nl} (v, e_s)^2 \quad \square$$

LEMMA 2.2. *If $v \in X_0$ and $(v_j)_j \subset C_{co}^\infty(\Omega_\delta)$ is such that $B(v - v_j, v - v_j) \rightarrow 0$, then*

1. $v_j \rightarrow v$ in $L^2(\Omega_\delta)$ (at least for a subsequence of $(v_j)_j$)
- 2.

$$\lim_j \sum_{k=1}^{\infty} \lambda_k^{nl} (v_{jk}) g_k = \sum_{k=1}^{\infty} \lambda_k^{nl} v_k g_k$$

for any $(g_k)_k \in V$, where $v_{jk} = (v_j, e_k)_{L^2(\Omega) \times L^2(\Omega)}$ (see the appendix for the definition of the Hilbert space V).

- 3.

$$(2.18) \quad \sum_{k=1}^{\infty} \lambda_k^{nl} (v, e_k)^2 \leq B(v, v).$$

Proof. From the hypothesis $\lim_j B(v_j, v_j) = B(v, v)$, and thus, from (2.17) we are led to write

$$\lim_j \sum_{k=1}^{\infty} \lambda_k^{nl} (v_{jk})^2 = B(v, v) < +\infty.$$

By Proposition 2.1, we know $\lambda_k^{nl} \rightarrow +\infty$ if $k \rightarrow +\infty$. Thus, the above equality implies that the sequence $\{(v_{jk})_k\}_j$ is uniformly bounded in V , and therefore, by Proposition 6.1, we can extract a subsequence strongly convergent in ℓ^2 to a sequence $(\tilde{v}_k)_k \in \ell^2$. (See the appendix for definitions of the Hilbert spaces ℓ^2 and V .) This is equivalent

to saying $v_j \rightarrow \tilde{v}$ in $L^2(\Omega_\delta)$, where $\tilde{v}$ is an element of $L_0^2(\Omega_\delta)$ defined by means of its Fourier coefficients $\tilde{v}_k$. Besides, it is clear that $(\tilde{v}_k)_k \in V$.

To see $\tilde{v} = v$, we notice $v_j - v \rightarrow \tilde{v} - v$ in $L^2(\Omega)$ and consequently, the lower semicontinuity of the quadratic form defined by B gives

$$(2.19) \quad B(\tilde{v} - v, \tilde{v} - v) \leq \lim_j B(v_j - v, v_j - v) = 0.$$

We also realize $\tilde{v} - v \in L_0^2(\Omega_\delta)$, which jointly with (2.19) implies $\tilde{v} - v = 0$ in Ω_δ and $v_j \rightarrow v$ in $L^2(\Omega)$.

Moreover, since $\{(v_{jk})_k\}_j$ is bounded in V , then it converges weakly in V to some sequence $(v_k^*)_k \in V$, that is,

$$(2.20) \quad \lim_j \sum_{k=1}^{\infty} \lambda_k^{nl} (v_{jk}) g_k = \sum_{k=1}^{\infty} \lambda_k^{nl} v_k^* g_k$$

for any $(g_k)_k \in V$. Take now any $(g_k)_k \in \ell^2$; clearly, if $G_k = \frac{g_k}{\lambda_k^{nl}}$, then $(G_k)_k \in V$ and

$$\lim_j \sum_{k=1}^{\infty} \lambda_k^{nl} (v_{jk}) G_k = \sum_{k=1}^{\infty} \lambda_k^{nl} v_k^* G_k.$$

But this reads as

$$\sum_{k=1}^{\infty} v_k g_k = \lim_j \sum_{k=1}^{\infty} v_{jk} g_k = \sum_{k=1}^{\infty} v_k^* g_k.$$

Therefore, $v^* = v$ (where v^* is the function defined by means of the sequence $(v_k^*)_k$) and (2.20) coincides with

$$(2.21) \quad \lim_j \sum_{k=1}^{\infty} \lambda_k^{nl} (v_{jk}) g_k = \sum_{k=1}^{\infty} \lambda_k^{nl} v_k g_k.$$

Equation (2.18) is an immediate consequence of the convergence $v_j \rightarrow v$ in $L^2(\Omega_\delta)$. $\square$

Proof (continuation of the proof Theorem 2.1). We turn to the proof of Theorem 2.1.

1. First, we consider $v \in C_{co}^\infty(\Omega_\delta)$ and $u \in X_0$. In such a case, we can take $(u_j)_j \subset C_{co}^\infty(\Omega_\delta)$ such that $B(u - u_j, u - u_j) \rightarrow 0$. By looking through the proof of Lemma 2.2, we know that $\{(u_{jk})_k\}_j$ converges weakly in V to $(u_k)_k$ when $j \rightarrow \infty$. Thus, since $(v, e_k)_k \in V$

$$B(v, u) = \lim_j B(v, u_j) = \lim_j \sum_{k=1}^{\infty} \lambda_k^{nl} (u_j, e_k) (v, e_k) = \sum_{k=1}^{\infty} \lambda_k^{nl} (u, e_k) (v, e_k).$$

We have just proved the following: if $v \in C_{co}^\infty(\Omega_\delta)$ and $u \in X_0$, then

$$B(v, u) = \sum_{k=1}^{\infty} \lambda_k^{nl} (u, e_k) (v, e_k).$$

2. The above statement can be extended to the case where both u and v belong to X_0 . Take $(u_j)_j$ and $(v_j)_j$ from $C_{co}^\infty(\Omega_\delta)$ such that $B(u - u_j, u - u_j) \rightarrow 0$ and $B(v - v_j, v - v_j) \rightarrow 0$ if $j \rightarrow +\infty$. Then the previous steps provide the equality we pursue: we observe that

$$B(u, v) = \lim_{j \rightarrow +\infty} B(u_j, v) = \lim_{j \rightarrow +\infty} \sum_{s=1}^{\infty} \lambda_s^{nl} (v, e_s) (u_j, e_s).$$

By the strong convergence of $\{(v_j, e_s)_s\}_j$ in ℓ^2 to $(v, e_s)_s$ and the Fatou lemma, we see that the sequence $(v, e_s)_s$ belongs to V . Also, we know $\{(u_j, e_s)_s\}_j \in V$ is uniformly bounded in V and it converges weakly to u in V . This implies

$$B(u, v) = \lim_{j \rightarrow +\infty} \sum_{s=1}^{\infty} \lambda_s^{nl} (v, e_s) (u_j, e_s) = \sum_{s=1}^{\infty} \lambda_s^{nl} (v, e_s) (u, e_s).$$

Also, by putting $u = v$, we can repeat the process to obtain formula (2.8) for any couple $u, v \in X_0$. This completes the proof of Theorem 2.1. $\square$

PROPOSITION 2.2 (see [36], [4]). *There is a positive constant C such that*

$$C \|w\|_{L^2(\Omega_\delta)}^2 \leq B(w, w)$$

for any $w \in X_0$.

Proof. See Proposition 2.1 and formula (2.8). $\square$

PROPOSITION 2.3. X_0 equipped with the scalar product $(f, g) = B(f, g)$ is a Hilbert space.

Proof. From Theorem 2.1

$$B(u, v) = \sum_{k=1}^{\infty} \lambda_k^{nl} u_k v_k$$

for any $u, v \in X_0$. This formula defines an inner product in X_0 . The proof of this fact is automatic. The only issue we have to check is the completeness: let $(f_n)_n$ be a Cauchy sequence in X_0 , that is,

$$B(f_n - f_m, f_n - f_m) \rightarrow 0$$

if $n, m \rightarrow +\infty$ or $\lim_{n, m \rightarrow +\infty} \sum_{k=1}^{\infty} \lambda_k^{nl} d_k^2 = 0$, where $d_k = c_k(n) - c_k(m)$ and $c_k(n) = \int_{\Omega_\delta} e_k f_n dx$. By Proposition 2.2, $(f_n)_n$ is a Cauchy sequence in $L_0^2(\Omega_\delta)$, and thus there is $g \in L_0^2(\Omega_\delta)$ such that $f_n \rightarrow g$ in $L_0^2(\Omega_\delta)$. Also, the convergence $B(f_n - f_m, f_n - f_m) \rightarrow 0$ is equivalent to state that the sequence of functions

$$G_n(x', x) = \frac{k_\delta^{1/2} (|x' - x|)}{|x' - x|} (f'_n - f_n)$$

is a Cauchy sequence of functions in $L^2(\Omega_\delta \times \Omega_\delta)$. Thus, we claim the existence of a function $G \in L^2(\Omega_\delta \times \Omega_\delta)$ such that $G_n \rightarrow G$ in $L^2(\Omega_\delta \times \Omega_\delta)$. If we take into account that at least for a subsequence

$$G_n(x', x) = \frac{k_\delta^{1/2} (|x' - x|)}{|x' - x|} (f'_n - f_n) \rightarrow \frac{k_\delta^{1/2} (|x' - x|)}{|x' - x|} (g' - g) \text{ a.e. in } \Omega_\delta \times \Omega_\delta,$$

then, necessarily, $G_n \rightarrow \frac{k_\delta^{1/2}(|x'-x|)}{|x'-x|}(g' - g)$ in $L^2(\Omega_\delta \times \Omega_\delta)$. This convergence can be expressed as

$$\lim_{n \rightarrow +\infty} \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{k_\delta(|x' - x|)}{|x' - x|^2} (f'_n - g' - (f_n - g))^2 dx' dx = 0$$

or, equivalently,

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^{\infty} \lambda_k^{nl} (D_k(n))^2 = 0,$$

where $D_k(n) = c_k(n) - g_k$. The two above equalities are equivalent to saying that the sequence of functions $(f_n)_n$ strongly converges to g in X_0 . $\square$

Remark 2.2. All the results from this section remain valid if we take $\Omega = Q$.

3. Spectrum of B_h .

Remark 3.1. Let $X_0(1)$ be the space X_0 defined by taking $h = 1$ and $X_0(h)$, the corresponding X_0 defined with any admissible h . Since $w = 0$ in $\partial\Omega^{nl}$ for any $w \in X_0(1)$ and the inequality $h \leq h_{\max}$ holds in Ω , then the decomposition

$$\Omega_\delta \times \Omega_\delta = (\Omega \times \Omega) \cup (\Omega \times (\Omega_\delta - \Omega)) \cup ((\Omega_\delta - \Omega) \times \Omega) \cup ((\Omega_\delta - \Omega) \times (\Omega_\delta - \Omega))$$

leads us to the inequality $B_h(w, w) < h_{\max} B(w, w)$ for any $w \in X_0(1)$. This means $X_0(1) \subset X_0(h)$. Let w be any element of $X_0(h)$; then there exists a sequence $(w_j)_j \in C_{co}^\infty(\Omega_\delta)$ such that $\lim_j B_h(w_j, w_j) = B_h(w, w)$. Moreover, it is enough to repeat the arguments from Lemma 2.2 to ensure that $(w_j)_j$ converges to w strong in $L^2(\Omega_\delta)$ (at least for a subsequence). By using, again, the above $\Omega_\delta \times \Omega_\delta$ -decomposition, the inequality $h \geq h_{\min}$ in Ω , and Proposition 2.2 we get

$$B_h(w_j, w_j) \geq \frac{h_{\min}}{2} B_1(w_j, w_j) \geq \frac{h_{\min}}{2} C \|w_j\|_{L^2(\Omega)}^2,$$

and therefore $B_h(w, w) \geq \frac{h_{\min}}{2} C \|w\|_{L^2(\Omega)}^2$. In particular, we have proved $X_0(h) \subset X_0(1)$.

From the above discussion, it is evident that there are several results to be drawn: for any $h \in L^\infty(\Omega_\delta; [h_{\min}, h_{\max}])$ and any $w \in X_0$, we have the double inequality

$$(3.1) \quad \frac{h_{\min}}{2} B(w, w) \leq B_h(w, w) \leq h_{\max} B(w, w).$$

Hence, the normed space X_0 is equivalent (in norm) to the same space but with $h = 1$. Thus, X_0 is a Hilbert space for any h in the space $L^\infty(\Omega_\delta; [h_{\min}, h_{\max}])$, (factually, X_0 does not change with h .) There are some other meaningful issues when we are dealing with the general case B_h , namely, (i) Lemma 2.2 item 1 is true and (ii) the L^2 -bound of Proposition 2.2 is still valid because by (3.1), we have $C \frac{h_{\min}}{2} \|w\|_{L^2(\Omega_\delta)}^2 \leq B_h(w, w)$. The remaining details of the proof of these statements follow along the same lines as section 2 and they are left to the reader.

Remark 3.2. Theorem 1.1 is going to be proved under the hypotheses on the kernel given by (1.2)–(1.5). Nevertheless, the assumptions (1.4) and (1.5) could be replaced by the classical bound

$$k_\delta(|s|) \geq \frac{C}{|s|^{N+2s-2}},$$

where C is a constant (probably depending on δ), $N > 2s$, and $s \in (0, 1)$. If we assume this new bound, we could include the admissible elements into the fractional Sobolev space $W^{s,2}(\Omega)$ and this fact will serve to obtain the needed relative compactness in L^2 for any bounded sequence in X_0 .

Proof of Theorem 1.1. We shall only sketch the proof because most of the details can be found in [33].

We study the minimization principle that gives rise to the first nonlocal eigenvalue:

$$(3.2) \quad \min_{w \in X_0: \|w\|_{L^2_0(\Omega_\delta)}=1} B_h(w, w).$$

Let us take an admissible minimizing sequence $(w_k^{(1)})_k$ in $X^{(1)} \doteq X_0$. Then there is a positive constant C such that $B_h(w_k^{(1)}, w_k^{(1)}) \leq C$ for any k . This bound and Proposition 2.2 give rise to

$$c \|w_k^{(1)}\|_{L^2(\Omega_\delta)}^2 \leq B_h(w_k^{(1)}, w_k^{(1)}) \leq C,$$

where c is a positive constant. Thus, we can take from $(w_k^{(1)})_k$ a subsequence weakly convergent to some $w_\delta^{(1)}$ in $L^2(\Omega_\delta)$. Besides, the weak lower semicontinuity of the quadratic form defined by B_h guarantees

$$B_h(w_\delta^{(1)}, w_\delta^{(1)}) \leq \liminf_k B_h(w_k^{(1)}, w_k^{(1)}) \leq C$$

and consequently $w_\delta^{(1)} \in X$. In order to show $w_\delta^{(1)} \in X_0$ and conclude the proof, it remains to verify $\|w_\delta^{(1)}\|_{L^2(\Omega_\delta)}^2 = 1$.

Consider now the Hilbert basis $(e_n)_n$ described in the previous section. Recall that the functions e_n , $n \geq 1$, are the eigenfunctions of the $-\Delta$ operator in Q . By applying the above inequality, (3.1), and Theorem 2.1 we get

$$(3.3) \quad C \geq B_h(w_k^{(1)}, w_k^{(1)}) \geq \frac{h_{\min}}{2} B(w_k^{(1)}, w_k^{(1)}) = C \sum_{n=1}^{\infty} \lambda_n^{nl} (w_{kn})^2,$$

where $w_{kn} = (w_k^{(1)}, e_n)_{L^2(\Omega) \times L^2(\Omega)}$. Then the sequence $\{(w_{kn})_n\}_k \in V$, thanks to (3.3), clearly satisfies

$$\sum_{n=1}^{\infty} (w_{kn})^2 \lambda_n^{nl} \leq C$$

uniformly in k . Since $(w_{kn})_n$ is a sequence in ℓ^2 and $\lambda_n^{nl} \rightarrow +\infty$ if $n \rightarrow +\infty$, we ensure the existence of a subsequence of $\{(w_{kn})_n\}_k$ strongly convergent in ℓ^2 (Proposition 6.1). This statement is equivalent to stating that, at least for a subsequence in k , $(w_k^{(1)})_k$ converges strongly in $L^2(\Omega)$ to $w_\delta^{(1)}$. We have just proved $\|w_\delta^{(1)}\|_{L^2(\Omega_\delta)}^2 = 1$, and therefore we claim the existence of a solution for (3.2).

In order to check the existence of the corresponding eigenvalue for the nonlocal functional, the procedure follows as in the classical setting: we take $\epsilon \in (-r, r)$ and

consider the perturbation $w^\epsilon = \frac{w_\delta^{(1)} + \epsilon\phi}{\|w_\delta^{(1)} + \epsilon\phi\|_{L^2}}$, $\phi \in X_0$. Then

$$\begin{aligned} B_h(w^\epsilon, w^\epsilon) &= \frac{B_h(w_\delta^{(1)} + \epsilon\phi, w_\delta^{(1)} + \epsilon\phi)}{\|w_\delta^{(1)} + \epsilon\phi\|_{L^2}^2} \\ &= \frac{B_h(w_\delta^{(1)}, w_\delta^{(1)}) + 2\epsilon B_h(w_\delta^{(1)}, \phi) + o(\epsilon)}{\|w_\delta^{(1)}\|_{L^2}^2 + 2\epsilon (w_\delta^{(1)}, \phi)_{L^2 \times L^2} + o(\epsilon)} \end{aligned}$$

and

$$\left. \frac{d}{d\epsilon} B_h(w^\epsilon, w^\epsilon) \right|_{\epsilon=0} = 0.$$

The above identity reads as

$$B_h(w_\delta^{(1)}, \phi) = B_h(w_\delta^{(1)}, w_\delta^{(1)}) (w_\delta^{(1)}, \phi)_{L^2 \times L^2},$$

which is equivalent to saying $w_\delta^{(1)}$ is an eigenvector whose corresponding eigenvalue is

$$\gamma_1^{nl}(\delta) = B_h(w_\delta^{(1)}, w_\delta^{(1)}) = \min_{X^{(1)}: \|w\|_{L_0^2(\Omega_\delta)}=1} B_h(w, w).$$

To prove $w_\delta^{(1)}$ is in X_0 , we note

$$\lim_k B_h(w_k^{(1)}, w_k^{(1)}) = B_h(w_\delta^{(1)}, w_\delta^{(1)})$$

and

$$\begin{aligned} \lim_k B_h(w_k^{(1)}, w_\delta^{(1)}) &= \lim_k \gamma_1^{nl}(\delta) (w_k^{(1)}, w_\delta^{(1)}) \\ &= \gamma_1^{nl}(\delta) \\ &= B_h(w_\delta^{(1)}, w_\delta^{(1)}). \end{aligned}$$

From the two above limits, we clearly conclude $w_k^{(1)} \rightarrow w_\delta^{(1)}$ in X_0 .

Concerning the remaining eigenfunctions, we can state the following: for each $n > 1$, there exists at least an eigenvector $w_\delta^{(n)}$ whose corresponding eigenvalue is $\gamma_n^{nl}(\delta)$ and

$$\gamma_n^{nl}(\delta) = B_h(w_\delta^{(n)}, w_\delta^{(n)}) = \min_{w \in X^{(n)}: \|w\|_{L_0^2(\Omega_\delta)}=1} B_h(w, w),$$

where

$$X^{(n)} = \left\{ u \in X_0 : B_h(u, w_\delta^{(j)}) = 0, \text{ for any } j = 1, \dots, n-1 \right\}.$$

As we have already commented, the remaining details come along with the arguments from [33]. $\square$

Remark 3.3. If we take $\Omega = Q$, then under the assumptions (1.2)–(1.5) the trigonometric system $(e_k)_k$ fulfills all the theses from Theorem 1.1.

4. Proof of Theorem 1.2. If $f \in L^2(\Omega)$, then by Theorem 1.1 it can be expanded in the orthonormal basis $(w_\delta^{(k)})_k$ of $L_0^2(\Omega_\delta)$, so that $f(x) = \sum_{k=1}^\infty f_{\delta k} w_\delta^{(k)}(x)$, where $(f_{\delta k})_k$ is the sequence of Fourier coefficients given by the formula $f_{\delta k} = \int_\Omega f(x) w_\delta^{(k)}(x) dx$.

We are going to show that the function

$$(4.1) \quad u_\delta(x) = \sum_{k=1}^\infty \frac{f_{\delta k}}{\gamma_k^{nl}} w_\delta^{(k)}(x)$$

is the solution to the problem. Indeed, since the $w_\delta^{(k)}$ are eigenfunctions for the nonlocal operator, it is clear that

$$(4.2) \quad B_h(u_\delta^M, v) = (f^M, v)_{L^2(\Omega) \times L^2(\Omega)}$$

for any $v \in X_0$, where $u_\delta^M = \sum_{k=1}^M \frac{f_{\delta k}}{\gamma_k^{nl}} w_\delta^{(k)}$ and $f^M = \sum_{k=1}^M f_{\delta k} w_\delta^{(k)}$. From the definition of u_δ^M is easy to check $u_\delta^M \rightarrow u_\delta$ in $L_0^2(\Omega_\delta)$. Moreover, $u_\delta^M \rightarrow u_\delta$ in X : indeed, since

$$u_\delta(x) - u_\delta^M(x) = \lim_{N \rightarrow +\infty} \sum_{k=M+1}^N \frac{f_{\delta k}}{\gamma_k^{nl}} w_\delta^{(k)}(x),$$

we can use the Fatou lemma to get

$$B_h(u_\delta - u_\delta^M, u_\delta - u_\delta^M) \leq \sum_{k=M+1}^\infty \gamma_k^{nl} \left(\frac{f_{\delta k}}{\gamma_k^{nl}} \right)^2,$$

and hence

$$\begin{aligned} \lim_{M \rightarrow +\infty} B_h(u_\delta - u_\delta^M, u_\delta - u_\delta^M) &\leq \lim_{M \rightarrow +\infty} \sum_{k=M+1}^\infty \gamma_k^{nl} \left(\frac{f_{\delta k}}{\gamma_k^{nl}} \right)^2 \\ &< C \lim_{M \rightarrow +\infty} \sum_{k=M+1}^\infty f_{\delta k}^2 = 0, \end{aligned}$$

where C is a positive constant. The lines from above determine $u_\delta \in X_0$.

If we take any $v = w_\delta^{(s)}$ in the (4.2) and take limits in $M \rightarrow \infty$, we get

$$(4.3) \quad B_h(u_\delta, w_\delta^{(s)}) = (f, w_\delta^{(s)}).$$

To prove (4.3), we employ the fact that $w_\delta^{(s)}$ is an eigenfunction and $u_\delta \in X_0$: taking limits on the right and left hands of (4.2) with $v = w_\delta^{(s)}$, we get $(f, w_\delta^{(s)})$ and

$$\lim_{M \rightarrow +\infty} B_h(u_\delta^M, w_\delta^{(s)}) = \lim_{M \rightarrow +\infty} \gamma_s^{nl} (u_\delta^M, w_\delta^{(s)}) = \gamma_s^{nl} (u_\delta, w_\delta^{(s)}) = B_h(u_\delta, w_\delta^{(s)}),$$

respectively. The equality (4.3) means that u_δ is the solution of (P_δ) .

The proof of unicity is trivial.

5. Proofs of Theorems 1.3 and 1.4.

5.1. Theorem 1.3. We split the proof into two parts.

1. Concerning the eigenfunction $w_\delta^{(1)}$, we know

$$\gamma_1^{nl}(\delta) = B_h(w_\delta^{(1)}, w_\delta^{(1)}) \leq B_h(\psi_1, \psi_1),$$

where ψ_1 is the first eigenfunction of the local problem on Ω with eigenvalue γ_1 (that is, ψ_1 and γ_1 verify (2.2) with $d = 0$). We assume that ψ_1 is zero outside of Ω and therefore ψ_1 is admissible for the minimizing principle defining the first nonlocal eigenvalue (see (6.1) with Ω_δ instead Ω). Then

$$(5.1) \quad \lim_{\delta \rightarrow 0} \gamma_1^{nl}(\delta) = \lim_{\delta \rightarrow 0} B_h(w_\delta^{(1)}, w_\delta^{(1)}) \leq \lim_{\delta \rightarrow 0} B_h(\psi_1, \psi_1) = \gamma_1,$$

where γ_1 is the first local eigenvalue that can be computed via the formula

$$(5.2) \quad \lim_{\delta \rightarrow 0} B_h(\psi_1, \psi_1) = b_h(\psi_1, \psi_1) = \gamma_1$$

and b_h is the bilinear form defined through the expression

$$b_h(\psi, \varphi) \doteq \int_{\Omega} h(x) (\nabla \psi(x) \cdot \nabla \varphi(x)) dx, \quad \psi, \varphi \in H^1(\Omega).$$

For the computation (5.2), we have written $B_h(\psi_1, \psi_1)$ as

$$\begin{aligned} B_h(\psi_1, \psi_1) &= \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x' - x|) H \frac{(\psi_1(x') - \psi_1(x))^2}{|x' - x|^2} dx' dx \\ &= \int_{\Omega} h(x) \int_{B(x, \delta)} \left(k_\delta(|x' - x|) \frac{(\psi_1(x') - \psi_1(x))^2}{|x' - x|^2} dx' \right) dx, \end{aligned}$$

and then we have used Corollary 1 from [10]. Thus, (5.1) allows us to assume the existence of a number $\delta_0 > 0$ such that

$$(5.3) \quad \frac{h_{\min}}{2} \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{k_\delta(|x' - x|)}{|x' - x|^2} \left(w_\delta^{(1)}(x') - w_\delta^{(1)}(x) \right)^2 dx' dx \leq B_h(w_\delta^{(1)}, w_\delta^{(1)}) \leq \gamma_1$$

for any $\delta < \delta_0$. Then $(w_\delta^{(1)})_\delta$ is bounded in $L^2(\Omega)$. According to Proposition 6.1 and (5.3), the sequence $(w_\delta^{(1)})_\delta$ is relatively compact in $L^2(\Omega)$. Then there exists $w^{(1)} \in L_0^2(\Omega_\delta)$ and a subsequence still denoted by $(w_\delta^{(1)})_\delta$ such that $w_\delta^{(1)} \rightarrow w^{(1)}$ in $L_0^2(\Omega)$ if $\delta \rightarrow 0$. From this convergence and part 2 of Theorem 6.1 from section 6, we can ensure $w^{(1)} \in H_0^1(\Omega)$.

At this point we shall assume $h(x) = \sum_{i=1}^m h_i I_{B_i}(x)$,² where $\{B_i\}$ is a finite covering of disjoint measurable sets of Ω and $(h_i)_{i=1}^m$ is a finite and bounded set

²Now $H(x', x) = \frac{\sum_{i=1}^m h_i I_{B_i}(x) + \sum_{i=1}^m h_i I_{B_i}(x')}{2}$.

of positive numbers such that $h_{\min} \leq h_i \leq h_{\max}$. Under these circumstances and taking into account how the support of k_δ is within $\Omega \times \Omega$, we can write

$$\begin{aligned} \gamma_1^{nl}(\delta) &= B_h(w_\delta^{(1)}, w_\delta^{(1)}) \\ &= \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x' - x|) H \frac{(w_\delta^{(1)}(x') - w_\delta^{(1)}(x))^2}{|x' - x|^2} dx' dx \\ &\geq \sum_{i=1}^m \int_{B_i} \int_{B_i} k_\delta(|x' - x|) \frac{h(x) + h(x')}{2} \frac{(w_\delta^{(1)}(x') - w_\delta^{(1)}(x))^2}{|x' - x|^2} dx' dx \\ &= \sum_{i=1}^m h_i \int_{B_i} \int_{B_i} k_\delta(|x' - x|) \frac{(w_\delta^{(1)}(x') - w_\delta^{(1)}(x))^2}{|x' - x|^2} dx' dx. \end{aligned}$$

That is,

$$(5.4) \quad \gamma_1^{nl}(\delta) = B_h(w_\delta^{(1)}, w_\delta^{(1)}) \geq \sum_{i=1}^m h_i I_i,$$

where

$$I_i = \int_{B_i} \int_{B_i} k_\delta(|x' - x|) \frac{(w_\delta^{(1)}(x') - w_\delta^{(1)}(x))^2}{|x' - x|^2} dx' dx.$$

The procedure exhibited by Ponce in [25, p. 12] and [26, Lemma 4, p. 245] allows us to write

$$\lim_{\delta \rightarrow 0} I_i \geq \int_{B_i} |\nabla w^{(1)}(x)|^2 dx.$$

Then, if we pass to the limit in (5.4), we get

$$\begin{aligned} \lim_{\delta \rightarrow 0} \gamma_1^{nl}(\delta) &= \lim_{\delta \rightarrow 0} B_h(w_\delta^{(1)}, w_\delta^{(1)}) \geq \lim_{\delta \rightarrow 0} \sum_{i=1}^m h_i I_i \\ &= \sum_{i=1}^m h_i \left(\lim_{\delta \rightarrow 0} I_i \right) \geq \sum_{i=1}^m h_i \int_{B_i} |\nabla w^{(1)}(x)|^2 dx \\ &= \int_{\Omega} h(x) |\nabla w^{(1)}(x)|^2 dx = b_h(w^{(1)}, w^{(1)}). \end{aligned}$$

Obviously, this fact and (5.1) are enough to conclude $w^{(1)} = \psi_1$, $b_h(w^{(1)}, w^{(1)}) = \gamma_1$, and $\lim_{\delta \rightarrow 0} \gamma_1^{nl}(\delta) = \gamma_1$.

Remark 5.1. Notice that we have just proved

$$\lim_{\delta} B_h(w_\delta^{(1)}, w_\delta^{(1)}) \geq \int_{\Omega} h(x) |\nabla w^{(1)}(x)|^2 dx$$

if h is a simple function.

If h is any measurable bounded function, we can find simple functions s_n such that $s_n(x) \rightarrow h(x)$ if $n \rightarrow +\infty$ for any $x \in \Omega$ and $0 \leq s_1 \leq s_2 \leq \dots \leq h$. Then it is straightforward to see that

$$\begin{aligned} \lim_{\delta} B_h(w_{\delta}^{(1)}, w_{\delta}^{(1)}) &\geq \lim_{\delta} B_{s_n}(w_{\delta}^{(1)}, w_{\delta}^{(1)}) \\ &\geq \int_{\Omega} s_n(x) \left| \nabla w^{(1)}(x) \right|^2 dx, \end{aligned}$$

and finally, letting $n \rightarrow \infty$,

$$\lim_{\delta} B_h(w_{\delta}^{(1)}, w_{\delta}^{(1)}) \geq \int_{\Omega} h(x) \left| \nabla w^{(1)}(x) \right|^2 dx.$$

2. Concerning the remainder eigenfunctions, for $k \geq 2$ we shall denote $(w_{i\delta}^{(k)})_{i=1}^{n_k}$ the eigenfunctions with eigenvalues $\gamma_k^{nl}(\delta)$.

We fix $k = 2$; clearly we cannot repeat the argument we use for $k = 1$ because in general

$$\gamma_2^{nl}(\delta) = B_h(w_{i\delta}^{(2)}, w_{i\delta}^{(2)}) \leq B_h(\psi_2, \psi_2)$$

is false. Nevertheless, the inequality

$$(5.5) \quad \lim_{\delta \rightarrow 0} \gamma_2^{nl}(\delta) = \lim_{\delta \rightarrow 0} B_h(w_{i\delta}^{(2)}, w_{i\delta}^{(2)}) \leq \gamma_2$$

will still be true. To check that we note

$$\gamma_2^{nl}(\delta) = B_h(w_{i\delta}^{(2)}, w_{i\delta}^{(2)}) \leq B_h(\zeta_{\delta}^{(2)}, \zeta_{\delta}^{(2)})$$

if $\zeta_{\delta}^{(2)}$ is any admissible function for the optimization problem defining the second nonlocal eigenvalue. We choose a sequence $\zeta_{\delta}^{(2)} = \alpha_1 w_{\delta}^{(1)} + \alpha_2 \psi_2$, fulfilling the requirements of admissibility:

$$\zeta_{\delta}^{(2)} = \alpha_1 w_{\delta}^{(1)} + \alpha_2 \psi_2 \in X^{(2)} \text{ and } \left\| \zeta_{\delta}^{(2)} \right\|_{L^2} = 1.$$

These conditions imply that the constants $\alpha_i = \alpha_i(\delta)$ have to be chosen so that

$$\left(\alpha_1 w_{\delta}^{(1)} + \alpha_2 \psi_2, w_{\delta}^{(1)} \right)_{L^2(\Omega) \times L^2(\Omega)} = 0 \text{ and } \alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 \left(w_{\delta}^{(1)}, \psi_2 \right) = 1.$$

Then, by denoting $w = (\psi_2, w_{\delta}^{(1)})_{L^2(\Omega) \times L^2(\Omega)}$ we have

$$\alpha_1 = -\alpha_2 w \text{ and } \alpha_2 = \pm \frac{1}{\sqrt{1-w^2}}.$$

By the previous analysis for $k = 1$, we get $\alpha_2 \rightarrow \pm 1$ and $\alpha_1 \rightarrow 0$, if $\delta \rightarrow 0$.

Then, by using again the first part, the bilinear form B_h upon the pair $(\zeta_{\delta}^{(2)}, \zeta_{\delta}^{(2)})$, in the limit, can be read as

$$\begin{aligned} \lim_{\delta \rightarrow 0} B_h(\zeta_{\delta}^{(2)}, \zeta_{\delta}^{(2)}) &= \lim_{\delta \rightarrow 0} \left(\alpha_1^2 B_h(w_{\delta}^{(1)}, w_{\delta}^{(1)}) \right. \\ &\quad \left. + \alpha_2^2 B_h(\psi_2, \psi_2) + 2\alpha_1\alpha_2 B_h(w_{\delta}^{(1)}, \psi_2) \right) \\ &= b_h(\psi_2, \psi_2) = \gamma_2, \end{aligned}$$

whence (5.5) is proved.

Now, with $k = 2$, we can follow the same procedure as in the case $k = 1$: the difference with respect to the case $k = 1$ stems from the fact that we could find more than one eigenvector. If we denote $(\psi_{i2})_{i=1}^{l_2}$ as the set of eigenfunctions for the local problem whose underlying eigenvalue is γ_2 , then we have seen $b_h(w_i^{(2)}, w_i^{(2)}) = \gamma_2$, $\lim_{\delta \rightarrow 0} \gamma_2^{nl}(\delta) = \gamma_2$ and we have proved the fact that the limits (in L^2)

$$(5.6) \quad \lim_{\delta \rightarrow 0} w_{i\delta}^{(2)} = w_i^{(2)}$$

are eigenfunctions of the local problem with eigenvalue γ_2 .

It remains to prove we attain all the local eigenfunctions; namely, we have to show the eigenfunctions we achieve, generate the same linear subspace in $H_0^1(\Omega)$ that the one obtained through the local eigenfunctions with eigenvalue γ_2 . In order to prove that we perform the following remarks:

(a) We can employ the same arguments from above to state

$$\lim_{\delta \rightarrow 0} \gamma_3^{nl}(\delta) = \lim_{\delta \rightarrow 0} B_h(w_{i\delta}^{(3)}, w_{i\delta}^{(3)}) \leq \gamma_3.$$

Indeed, we can choose $\zeta_\delta^{(3)} = \alpha_1 w_\delta^{(1)} + \sum_{i=1}^{n_2} \alpha_{2i} w_{i\delta}^{(2)} + \alpha_3 \psi_3 \in X^{(3)}$ such that $\|\zeta_\delta^{(3)}\|_{L^2} = 1$. Since ψ_3 is orthogonal to $w_\delta^{(1)}$ and to $w_i^{(2)}$ for any $i = 1, \dots, n_2$, then

$$(\psi_3, w_\delta^{(1)}) = 0, \quad \lim_{\delta \rightarrow 0} (\psi_3, w_{i\delta}^{(2)}) = 0, \quad i = 1, \dots, n_2.$$

These facts lead us to $\lim_{\delta \rightarrow 0} \gamma_3^{nl}(\delta) = b_h(w_i^{(3)}, w_i^{(3)}) = \gamma_3$, where $w_i^{(3)} = \lim_{\delta \rightarrow 0} (w_{i\delta}^{(3)})_\delta$. In general, for any k , we have

$$\lim_{\delta \rightarrow 0} \gamma_k^{nl}(\delta) = \lim_{\delta \rightarrow 0} B_h(w_{i\delta}^{(k)}, w_{i\delta}^{(k)}) \leq \gamma_k$$

and therefore $\lim_{\delta \rightarrow 0} \gamma_k^{nl}(\delta) = b_h(w_i^{(k)}, w_i^{(k)}) = \gamma_k$ being $w_i^{(k)} = \lim_{\delta \rightarrow 0} (w_{i\delta}^{(k)})_\delta$.

(b) Assume that $(\psi_{i2})_{i=1}^{l_2}$ is the set of eigenfunctions for the local problem whose underlying eigenvalue is γ_2 and consider the functions $(w_i^{(2)})_{i=1}^{n_2}$ defined via the limits (5.6). These functions are eigenfunctions, but we do not know whether $l_2 = n_2$. In either case, it is clear that the number of local eigenfunctions cannot be bigger than l_2 , $n_2 \leq l_2$. We are going to prove the equality. The proof is given by using a contradiction argument. First, we note

$$(5.7) \quad \gamma_3 = \lim_{\delta \rightarrow 0} \gamma_3^{nl}(\delta) > \lim_{\delta \rightarrow 0} \gamma_2^{nl}(\delta) = \gamma_2$$

Also, since we are assuming $n_2 < l_2$, there is at least one ψ from the set $(\psi_{i2})_{i=1}^{l_2}$ such that $\psi \notin \mathcal{L}\{(w_i^{(2)})_{i=1}^{n_2}\}$, so that

$$(\psi, w_\delta^{(1)}) = 0 \quad \text{and} \quad (\psi, w_i^{(2)}) = 0, \quad \text{for } i = 1, \dots, n_2$$

or

$$\lim_{\delta \rightarrow 0} (\psi, w_\delta^{(1)}) = 0 \quad \text{and} \quad \lim_{\delta \rightarrow 0} (\psi, w_{i\delta}^{(2)}) = 0 \quad \text{for } i = 1, \dots, n_2.$$

Besides, ψ can be written in the basis $(w_\delta^{(k)})_k$ for any δ : more precisely, ψ can be written in the basis

$$\cup_{k=1}^{\infty} \cup_{i=1}^{n_k} \left(w_{i\delta}^{(k)} \right)_k.$$

Thus, we infer

$$\psi = \sum_{k=1}^{\infty} \sum_{i=1}^{n_i} \psi_{i\delta k} w_{i\delta}^{(k)} \text{ and } \|\psi\|_{L^2}^2 = \sum_{k=1}^{\infty} \sum_{i=1}^{n_i} (\psi_{i\delta k})^2 = 1,$$

where $\psi_{i\delta k} = (\psi, w_{i\delta}^{(k)})$. These equalities can be manipulated to derive the identities

$$\begin{aligned} 1 = \|\psi\|_{L^2}^2 &= \lim_{\delta \rightarrow 0} \sum_{k=1}^2 \sum_{i=1}^{n_i} (\psi_{i\delta k})^2 + \lim_{\delta \rightarrow 0} \sum_{k=3}^{\infty} \sum_{i=1}^{n_i} (\psi_{i\delta k})^2 \\ (5.8) \quad &= \lim_{\delta \rightarrow 0} \sum_{k=3}^{\infty} \sum_{i=1}^{n_i} (\psi_{i\delta k})^2. \end{aligned}$$

Then, by using (5.7) and (5.8) we arrive at

$$\begin{aligned} \gamma_2 &= b_h(\psi, \psi) = \lim_{\delta \rightarrow 0} B_h(\psi, \psi) = \lim_{\delta \rightarrow 0} \sum_{k=1}^{\infty} \sum_{i=1}^{n_i} (\psi_{i\delta k})^2 \gamma_k^{nl}(\delta) \\ &= \lim_{\delta \rightarrow 0} \sum_{k=3}^{\infty} \sum_{i=1}^{n_i} (\psi_{i\delta k})^2 \gamma_k^{nl}(\delta) \\ &\geq \lim_{\delta \rightarrow 0} \gamma_3^{nl}(\delta) \lim_{\delta \rightarrow 0} \sum_{k=3}^{\infty} \sum_{i=1}^{n_i} (\psi_{i\delta k})^2 \\ &\geq \lim_{\delta \rightarrow 0} \gamma_3^{nl}(\delta) = \gamma_3, \end{aligned}$$

which is a contradiction. Then, $n_2 = l_2$ and we achieve the same number of eigenfunctions l_2 .

Finally, it is trivial to check that both linear subspaces, $\mathcal{L}\{(w_i^{(2)})_{i=1}^{n_2}\}$ and $\mathcal{L}\{(\psi_{i2})_{i=1}^{n_2}\}$, coincide.

3. Last, induction in k provides the proof of Theorem 1.3.

5.2. Theorem 1.4. We analyze the limit when $\delta \rightarrow 0$ of the solution of the problem (P_δ)

$$u_\delta = \sum_{k=1}^{\infty} \frac{f_{\delta k}}{B_h(w_\delta^{(k)}, w_\delta^{(k)})} w_\delta^{(k)}.$$

From the equation $B_h(u_\delta, \psi) = (f, \psi)$ for any $\psi \in X_0$, we get $B_h(u_\delta, u_\delta) \leq \|f\|_{L^2(\Omega)} \|u_\delta\|_{L^2(\Omega_\delta)}$. Additionally, by Proposition 2.2 we know a reverse bound; there is $C > 0$ independent of δ ($\delta \geq \delta_0$) such that $B_h(u_\delta, u_\delta) \geq C \|u_\delta\|_{L^2(\Omega_\delta)}^2$ (see 3.1). Thus, we get $B_h(u_\delta, u_\delta) \leq C$ and by Theorem 6.1 the sequence $(u_\delta)_\delta$ converges strongly in $L^2(\Omega)$ (at least for a subsequence) to a function $u \in H_0^1(\Omega)$. Consequently, if we write

$u = \sum u_k w^{(k)}$, then the strong convergence in L^2 of u_δ and the strong convergence in L^2 of $w_\delta^{(k)}$ (see Theorem 1.3) ensure

$$\lim_{\delta \rightarrow 0} (u_\delta)_k = \lim_{\delta \rightarrow 0} (u_\delta, w_\delta^{(k)}) = (u, w^{(k)}) = u_k.$$

But this enables us to state

$$u_k = \lim_{\delta \rightarrow 0} (u_\delta)_k = \lim_{\delta \rightarrow 0} \frac{f_{\delta k}}{B_h(w_\delta^{(k)}, w_\delta^{(k)})} = \frac{f_k}{\gamma_k},$$

where $f_k = (w^{(k)}, f) = \int_\Omega w^{(k)}(x) f(x) dx$. We get

$$u = \sum_{k=1}^{\infty} \frac{f_k}{\gamma_k} w^{(k)}.$$

Now, we check $b_h(u, \psi) = (f, \psi)$ for any $\psi \in H_0^1(\Omega)$: by putting $\psi = \sum \psi_k w^{(k)}$ we have

$$\begin{aligned} b_h(u, \psi) &= b_h\left(\sum_{k=1}^{\infty} \frac{f_k}{\gamma_k} w^{(k)}, \sum_{k=1}^{\infty} \psi_k w^{(k)}\right) \\ &= \sum_{k=1}^{\infty} \frac{f_k}{\gamma_k} \psi_k b_h(w^{(k)}, w^{(k)}) \\ &= \sum_{k=1}^{\infty} f_k \psi_k = (f, \psi), \end{aligned}$$

and therefore u , the strong limit of u_δ , is the solution of the local problem (P).

We conclude by remarking that the strong convergence in X_0 is automatic because

$$\lim_{\delta \rightarrow 0} B_h(u_\delta, u) = \lim_{\delta \rightarrow 0} (f, u) = b_h(u, u) = \lim_{\delta \rightarrow 0} B_h(u, u)$$

and

$$\lim_{\delta \rightarrow 0} B_h(u_\delta, u_\delta) = \lim_{\delta \rightarrow 0} (f, u_\delta) = (f, u) = b_h(u, u) = \lim_{\delta \rightarrow 0} B_h(u, u).$$

6. Appendix. Here we locate some basic results used throughout the paper.

PROPOSITION 6.1 (see [12]). *Let ℓ^2 be the set of sequences $(u_n)_n$ such that $\sum u_n^2 < \infty$ and consider $(\lambda_n)_n$ to be a sequence of positive numbers such that $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$. Let V be the space of sequences $(u_n)_n$ such that $\sum \lambda_n |u_n|^2 < \infty$ and assume the space V is equipped with the scalar product $((u_n), (v_n)) = \sum \lambda_n u_n v_n$. Then, V is a Hilbert space and $V \subset \ell^2$ with compact injection.*

PROPOSITION 6.2 (generalized Riemann–Lebesgue lemma; see [21]). *Let β be a function in $L^\infty(\mathbb{R}^N)$ such that the limits*

$$M_\beta(\omega) = \lim_{R \rightarrow +\infty} \frac{\int_0^R \beta(r\omega) dr}{R}$$

exist for almost all $\omega \in S^{N-1}$. Then,

$$\lim_{\lambda \rightarrow +\infty} \int_{\mathbb{R}^N} f(s) \beta(\lambda s) ds = \int_{\mathbb{R}^N} f(s) M_\beta\left(\frac{s}{|s|}\right) ds$$

for any $f \in L^1(\mathbb{R}^N)$.

PROPOSITION 6.3 (see [10], [11]). Assume $f \in H^1(\Omega)$ and $K \in L^1(\mathbb{R}^N)$, $K \geq 0$. Then,

$$(6.1) \quad \int_{\Omega} \int_{\Omega} K(|x' - x|) \frac{|f(x') - f(x)|^2}{|x' - x|^2} dx' dx \leq C \|\nabla f\|_{L^2}^2 \|K\|_{L^1},$$

where C only depends on Ω .

THEOREM 6.1 (see [25], [10], [1]). If $(f_{\delta})_{\delta}$ is a bounded sequence in $L_0^2(\Omega_{\delta})$ and there is a positive constant such that

$$B_h(f_{\delta}, f_{\delta}) \leq C$$

for any $\delta \leq \delta_0$ and δ_0 given (B_h defined as in (1.7)), then

1. $(f_{\delta})_{\delta}$ is relatively compact in $L^2(\Omega)$,
2. if $(f_{\delta_j})_j$ is a subsequence of $(f_{\delta})_{\delta}$ such that $f_{\delta_j} \rightarrow f$ in $L^2(\Omega)$, then $f \in H^1(\Omega)$.

Proof. Clearly, for any δ

$$\int_{\Omega} \int_{\Omega} k_{\delta}(|x' - x|) \frac{(f_{\delta}(x') - f_{\delta}(x))^2}{|x' - x|^2} dx' dx \leq \frac{C}{h_{\min}}$$

and these inequalities allow us to use Theorem 1.2 from [25] and conclude the proof. $\square$

PROPOSITION 6.4. Assume $n \in \mathbb{N}$ and $\Delta u = -\lambda u$, then

$$(6.2) \quad \begin{aligned} I_n &= \int_{B(0, \delta)} \frac{k(|s|)}{|s|^2} \left(s_1 \frac{\partial u(x)}{\partial x_1} + s_2 \frac{\partial u(x)}{\partial x_2} \right)^{2n} ds \\ &= \Delta^n u(x) \int_{B(0, \delta)} \frac{k(|s|)}{|s|^2} (s_1)^{2n} ds \\ &= (-1)^n u(x) \int_{B(0, \delta)} \frac{k(|s|)}{|s|^2} \left(\sqrt{\frac{\lambda}{2}} (s_1 + s_2) \right)^{2n} ds. \end{aligned}$$

Proof. We use the notation given in (2.15). It is clear that integrals vanish if k is even, and therefore we must compute

$$I_n = \int_{B(0, \delta)} \frac{k(|s|)}{|s|^2} \left(\sum_{j=0}^n \binom{2n}{2j} (s_1)^{2j} (s_2)^{2n-2j} \frac{\partial^{2n} u(x)}{\partial (x_1)^{2j} \partial (x_2)^{2n-2j}} \right) ds.$$

We carefully look at the integrals for $j \in \{0, 1, \dots, n\}$

$$I_n(j) = \int_{B(0, \delta)} \frac{k(|s|)}{|s|^2} \binom{2n}{2j} (s_1)^{2j} (s_2)^{2n-2j} \frac{\partial^{2n} u(x)}{\partial (x_1)^{2j} \partial (x_2)^{2n-2j}} ds,$$

which after changing to polar coordinates can be written as

$$\frac{\partial^{2n} u(x)}{\partial (x_1)^{2j} \partial (x_2)^{2n-2j}} \left(\int_0^{\delta} \frac{k(|r|)}{C_{\delta} |r|^2} r^{2n+1} dr \right) \binom{2n}{2j} \int_0^{2\pi} (\cos \theta)^{2j} (\sin \theta)^{2n-2j} d\theta.$$

Some formulae from elemental calculus are going to be employed:

1. Integration by parts³ and induction provides the proof of the equality

$$\int_0^{2\pi} \cos^{2j} \theta \sin^{2n-2j} \theta d\theta = \int_0^{2\pi} \sin^{2j} \theta \cos^{2n-2j} \theta d\theta$$

with $j \in \{0, 1, \dots, n-1\}$ and $n \in \mathbb{N}$

2. Also, an induction procedure provides

$$\int_0^{2\pi} \sin^2 \theta \cos^{2n-2} \theta d\theta = \frac{1}{2n-1} \int_0^{2\pi} \sin^{2n} \theta d\theta,$$

and more generally

$$\int_0^{2\pi} \sin^{2j} \theta \cos^{2n-2j} \theta d\theta = \frac{2j-1}{2n-(2j-1)} \int_0^{2\pi} \sin^{2j-2} \theta \cos^{2n-(2j-2)} \theta d\theta.$$

3. According to the above formulae, we arrive at

$$\begin{aligned} & \int_0^{2\pi} \sin^{2j} \theta \cos^{2n-2j} \theta d\theta \\ &= \frac{2j-1}{2n-(2j-1)} \times \frac{2j-3}{2n-(2j-3)} \times \dots \times \frac{3}{2n-3} \times \frac{1}{2n-1} \int_0^{2\pi} \sin^{2n} \theta d\theta, \end{aligned}$$

where $j \in \{1, 2, \dots, \frac{n}{2}\}$ if n is even and $j \in \{1, 2, \dots, \frac{n-1}{2}\}$ if n is odd.

These results and some easy computations allow us to write

$$\begin{aligned} I_n(j) &= \int_{B(0,\delta)} \frac{k(|s|)}{|s|^2} \binom{2n}{2j} (s_1)^{2j} (s_2)^{2n-2j} \frac{\partial^{2n} u(x)}{\partial (x_1)^{2j} \partial (x_2)^{2n-2j}} ds \\ &= \frac{\partial^{2n} u(x)}{\partial (x_1)^{2j} \partial (x_2)^{2n-2j}} \int_{B(0,\delta)} \frac{k(|s|)}{|s|^2} \binom{n}{j} (s_1)^{2n} ds. \end{aligned}$$

Then we can rewrite the initial integral as

$$\begin{aligned} I_n &= \sum_{j=0}^n I_n(j) = \sum_{j=0}^n \binom{n}{j} \frac{\partial^{2n} u(x)}{\partial (x_1)^{2j} \partial (x_2)^{2n-2j}} \int_{B(0,\delta)} \frac{k(|s|)}{|s|^2} (s_1)^{2n} ds \\ &= \Delta^n u(x) \int_{B(0,\delta)} \frac{k(|s|)}{|s|^2} s_1^{2n} ds = (-1)^n \lambda^n u(x) \int_{B(0,\delta)} \frac{k(|s|)}{|s|^2} s_1^{2n} ds, \end{aligned}$$

where the last equality is true because $\Delta^n u = (-1)^n \lambda^n u$. To finish, we look at the last integral of (6.2): concretely, we note that the only nonvanishing terms derived from $(\frac{s_1+s_2}{\sqrt{2}})^{2n}$ are even; they are $(\frac{1}{\sqrt{2}})^{2n} \sum_{k=1}^{2n} \binom{2n}{k} (s_1)^k (s_2)^{2n-k}$ with k even, and the integral containing the term $\binom{2n}{2j} (s_1)^{2j} (s_2)^{2n-2j}$ coincides with the one that contains $\binom{n}{j} (s_1)^{2n}$ (this fact has been derived in the discussion from above) and $\sum_{j=0}^n \binom{n}{j} = 2^n$. All these facts allow us to substitute $\int_{B(0,\delta)} \frac{k(|s|)}{|s|^2} s_1^{2n} ds$ by $\int_{B(0,\delta)} \frac{k(|s|)}{|s|^2} (\frac{s_1+s_2}{\sqrt{2}})^{2n} ds$. This finishes the proof. $\square$

³Integrating by parts, we can prove

$$\int_0^{2\pi} \sin^n \theta \cos^m \theta d\theta = \frac{m-1}{m+n} \int_0^{2\pi} \sin^n \theta \cos^{m-2} \theta d\theta \quad \int_0^{2\pi} \sin^{2n} \theta d\theta = \frac{2n-1}{2n} \int_0^{2\pi} \sin^{2n-2} \theta d\theta$$

and

$$\int_0^{2\pi} \cos^{2n} \theta d\theta = \frac{2n-1}{2n} \int_0^{2\pi} \cos^{2n-2} \theta d\theta.$$

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